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EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10

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NON-LINEAR VIBRATIONS: A CHAPTER IN MATHEMATICAL HISTORY.

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BY M. L. CARTWRIGHT.

SINCE about 1939 my researches have been, in collaboration with Professor J. E. Littlewood, mainly in a field of mathematics sometimes described as non-linear mechanics, but as most of the equations in mechanics are non-linear, it is better described as non-linear vibrations or oscillations. It is rather a curious branch of mathematics developed by different people from different standpoints, straight mechanics, radio oscillations, pure mathematics and servo-mechanisms of automatic control theory. The notation varies from equations in v for voltage to θ for the angle of the pendulum, R for the radius of a gas-filled cavity, and x and y of pure mathematicians, with parameters ranging through the Greek and Roman alphabets. The problems vary from practical problems which can be solved by fairly straightforward approximation by linear equations to highbrow problems in topology and the theory of sets of points. There is probably no one competent to treat the subject adequately in all these aspects, but we try to exchange problems, ideas and methods, and I propose to try to describe how it was developed, but my account will necessarily be somewhat biased by the circumstances of my introduction to it which I shall describe presently, and over-simplified on account of the limitations of time. The variety of notation has increased my difficulties of presentation.

It is usual to restrict the term non-linear vibrations to those parts of the subject in which the differential equations representing the phenomena of periodic or almost periodic oscillations are non-linear in the dependent variable and cannot be integrated by elementary methods nor by using standard tables for elliptic or Bessel functions. For example, the oscillations of a simple pendulum are represented by

$$\ddot{\theta}+\frac{g}{l}\sin\theta=0,$$

where dots denote differentiation with respect to t the time. This is non-linear in θ, but if we multiply by $\dot\theta$ and integrate we have

$$\tfrac{1}{2}\dot\theta^2 = n^2(\cos\alpha - \cos\theta), \quad n^2 = g/l,$$

where $\theta = \alpha$ when $\dot\theta = 0$. By putting $\sin\tfrac{1}{2}\theta = \sin\tfrac{1}{2}\alpha\sin\phi$ and various manipulations we can express t in the form

$$\frac{1}{n}\int_0^\phi \frac{d\phi}{(1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi)^{\frac{1}{2}}},$$

which is an elliptic integral. Then, by means of tables, we can read off the results required, but if we complicate the problem by inserting any form of damping, or by means of a forcing term, the problem becomes a characteristic problem of non-linear vibrations. For example, the subject is concerned with equations of which

$$\ddot\theta + 2f\dot\theta + n^2\sin\theta = 0$$

and

$$\ddot\theta + n^2\sin\theta = P\cos\lambda t$$

are special cases. So far as I know there is no special device for integrating these by elementary methods.

One of the earliest contributions to the theory was Duffing's study (published in Germany in 1918) of an approximation to the second type of equation

$$\ddot x + \alpha x + \beta x^3 = P\cos\lambda t,$$

but it was considered in connection with a spring of variable stiffness with restoring force $\alpha x + \beta x^3$ where $\alpha > 0$ and β could be positive or negative. The corresponding linear equation with $\beta = 0$ has a solution of period $2\pi/\lambda$ of the form $x = \dfrac{P}{\alpha - \lambda^2}\cos\lambda t$, with an amplitude $P/|\,\alpha - \lambda^2\,|$ which is symmetrical in $\alpha - \lambda^2$ and tends to ∞ as $\lambda \to \alpha^{\frac{1}{2}}$. From physical considerations Duffing knew that for the range of α, β, P, λ considered there was a solution of period $2\pi/\lambda$ of the form $x = A\cos\lambda t + x_1$, where x_1 is small, and so, arranging the trigonometrical terms as in a Fourier series so as to correspond to the physical vibrations, we have $(-\lambda^2 A + \alpha A + \tfrac{3}{4}\beta A^3 - P)\cos\lambda t + \tfrac{1}{4}\beta A^3\cos 3\lambda t +$ smaller terms. Since this holds over several periods $2\pi/\lambda$ at least, it suggests that in the first place β must be small (unless A itself is small) and that

$$(\alpha - \lambda^2)A + \tfrac{3}{4}\beta A^3 = P.$$

Various difficulties arise, but if we assume that $P = \beta P_0$ is also small, which is the interesting case, it can be shown that as λ^2 varies, A behaves in quite a different manner from that for the linear case. First, A does not tend to ∞ as $\lambda^2 \to \alpha$ but takes the value $(\tfrac{4}{3}P_0)^{\frac{1}{3}}$ and, more characteristically, it is unsymmetrical with respect to $\lambda^2 = \alpha$ and for some values of the parameters three different values of A may correspond to one value of λ. This is typical of quasi-linear problems in which the use of approximations to linear equations explain phenomena not seen in linear systems.

From a strictly mathematical point of view, perhaps the most important part of the *linear* theory is the principle of superposition. Consider the equation

$$\ddot x + F(t)\dot x + G(t)x = 0,$$

where F and G depend on t only. If $x = \phi_1(t)$, $x = \phi_2(t)$ are solutions, it is easy to verify that

$$x = A\phi_1(t) + B\phi_2(t)$$

is also a solution, and it can be shown that *any* solution can be expressed in

this form provided that ϕ_1, ϕ_2 are linearly independent. Further, if there is a forcing term $P(t)$ of period $2\pi/\lambda$ and $x=\psi(t)$ is a solution of the non-homogeneous equation,

$$\ddot{x}+F(t)\dot{x}+G(t)x=P(t),$$

then

$$x=\psi(t)+A\phi_1(t)+B\phi_2(t)$$

is a solution and any other solution of the non-homogeneous equation can be expressed in this way. By analogy with the well-known equation with constant coefficients which I shall write

$$\ddot{x}+2f\dot{x}+n^2x=P\cos\lambda t, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

I shall call $F(t)$ the damping, $G(t)x$ the restoring force ; $A\phi_1+B\phi_2$ is the free oscillation, and a solution $\psi(t)$ of period $2\pi/\lambda$ is the forced oscillation.

It is useful to recall that if $n>f$, the solution of (1) with $P=0$ is

$$x=ae^{-ft}\sin(\mu t+\alpha), \quad \text{where } \mu^2=n^2-f^2,$$

so that if $f>0$, the oscillations die down while if $f<0$, they increase indefinitely in amplitude. If $P\neq 0$ there is also a forced oscillation of period $2\pi/\lambda$, the amplitude of which has a maximum with respect to λ at $\lambda=\alpha^{\frac{1}{2}}$.

Much of the non-linear theory has been concerned with equations of the form

$$\ddot{x}+f(x)\dot{x}+g(x)=p(t),$$

where $p(t)$ has period $2\pi/\lambda$, and $g(x)$ behaves in a somewhat similar manner to a restoring force so that $g(x)/x>0$ at any rate for large $|x|$. The damping $f(x)$ (which sometimes depends on $\dot{x}$ as well) is usually positive for large $|x|$. These conditions are fairly typical of the physical systems considered. For such equations the principle of superposition does not hold, and we have already seen that the introduction of a forcing term can introduce new features.

One of the most famous equations in the theory is that introduced by van der Pol in 1920,

$$\ddot{x}-k(1-x^2)\dot{x}+x=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Here the damping changes sign. If k is small the solution over a limited time is approximately of the form $x=a\sin t$, but a may vary slowly. As might be

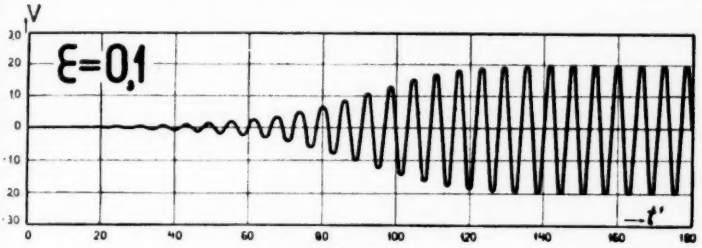

FIG. 1

This is one of van der Pol's solutions obtained by graphical methods for equation (2) written in the form

$$\ddot{v}-\epsilon(1-v^2)\dot{v}+v=0.$$

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expected from comparison with the linear case, a increases when it is small and decreases when it is large. Between these two cases, it can be shown there

is one in which a is more or less stationary, corresponding to a strictly periodic solution. In this case, both x and $\dot{x}$ return to their initial values after one complete period which is approximately 2π. We can find an approximation for this stationary value of a from the energy equation. Multiplying as usual by $\dot{x}$, integrating from 0 to 2π and remembering that x and $\dot{x}$ return to their initial values, we have approximately

$$\int_0^{2\pi} (1 - a^2 \sin^2 t) \cos^2 t \, dt = 0,$$

which gives $a = 2$. Although this is independent of k, our initial approximation $x = a \sin t$ is only valid for k small, and so the result is only valid for k small. It can in fact be shown that in this case the solutions are oscillations of three kinds. For one kind $a < 2$ and a increases slowly, for another $a > 2$ and a decreases slowly, and for the third type of solution a remains stationary and $a = 2$. The non-linearity may be said to control the amplitude in the sense that it allows it to increase when it is small but prevents it becoming too large.

The general solution cannot be obtained by the combination of two linearly independent solutions and similar difficulties arise when we add a forcing term to this equation. This was brought out very clearly by the work of van der Pol and Appleton, partly in collaboration, and partly independently, in a series of papers on radio oscillations published between 1920 and 1927. To me the work of the radio engineers is much more interesting and suggestive than that of the mechanical engineers. The radio engineers want their systems to oscillate, and to oscillate in a very orderly way, and therefore they want to know not only whether the system has a periodic solution, but whether it is stable, what its period and amplitude and harmonic content are, and how these vary with the parameters of the equation, and they sometimes want the period to be determined with a very small error. In the early days they wanted to explain why the amplitude was limited in a certain way and why in some cases the period lengthened as the harmonic content increased and not in others. The desire to know why and the insistence on how the various quantities such as amplitude and frequency vary with the parameters of the equation over fairly wide ranges meant that numerical and graphical solutions either failed to provide the answer or were far too cumbersome. Further, unless one knows something about the general behaviour of the solutions, the numerical work, which is only approximate, may be misleading.

The speed of radio oscillations has been an important factor in the development of the subject. Somewhat similar equations had been studied in connection with astronomy, but there the cycles are measured in days and years instead of fractions of a second and therefore have not the power to suggest the behaviour of stable solutions in the way that some of the radio work has done.

Consider, for instance, van der Pol's equation with forcing term

$$\ddot{x} - k(1 - x^2)\dot{x} + x = pk \cos \lambda t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

with k small and λ near 1. The damping term is small unless x or $\dot{x}$ is large, and if we ignore this possibility, since λ is near 1 the free and forced oscillations are both approximately of the form $a \sin(\lambda t + \alpha)$ over a time $10\pi/\lambda$, say. Appleton and van der Pol were able to observe that in the physical system which corresponded approximately the oscillations settled down almost at once to a steady state in which a and α had a long period. They varied slowly with the time except for λ in a small interval very near $\lambda = 1$. In this small interval what corresponds roughly to the free oscillation, that is to say the solution of the equation with $p = 0$, synchronized with the forced oscillation of

period $2\pi/\lambda$. This means that a and α were constant with respect to the time. The synchronization could be clearly detected by ear. Outside this interval of λ they were able to observe the variations in a and α and to see how the pattern of them changed as λ approached the interval of synchronization. For instance, away from this interval oscillograms showed a varying in a regular way rather like a sine curve, but near synchronization a seemed to

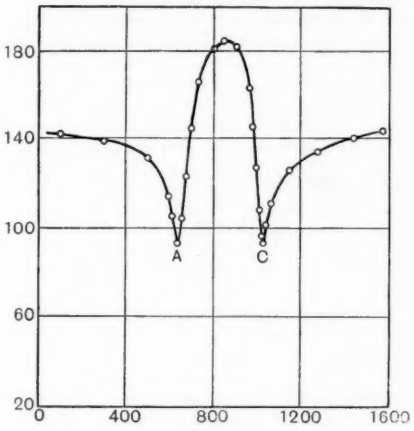

FIG. 2

This is the resonance curve obtained by Appleton for a triode oscillator when the frequency of a coupled transmitter was varied through resonance. The square of the oscillatory potential is exhibited as a function of the transmitter condenser readings.

Proc. Camb. Phil. Soc., XXI (1922), p. 242

linger near the value which it would take on synchronization and pass through the rest of its cycle of values more rapidly. The average value of a^2 could also be observed showing resonance curves which descended as λ approached the values corresponding to synchronization from outside. and then rising rapidly to a maximum near $\lambda = 1$.

The method of van der Pol consisted in putting

$$x = a_1 \cos \lambda t + a_2 \sin \lambda t$$

and supposing that $\dot{a}_1$, $\dot{a}_2$ were small and ignoring $\ddot{a}_1$, $\ddot{a}_2$. Guided by physical intuition, he obtained approximations which are fundamentally sound except perhaps for values of λ very near those at which synchronization takes place, and here the mathematical explanation of some of the phenomena seemed to defeat the radio engineers. The mathematical treatment of these approximations was rather casual, although Greaves provided a justification of those obtained by him and Appleton for a generalization of this equation in the special case $p = 0$.

At this point it may be worth while to recall something of the development of allied branches of mathematics. Poincaré in France, Liapounov in Russia and Birkhoff in America had been developing classical mechanics with the precision of a branch of pure mathematics, but in England a gulf had formed between the pure and applied mathematicians. This is shown by the remarks

of Sir Joseph Larmor in an address to the London Mathematical Society. He said " Progress might be assisted, with benefits not all on one side, by continuous and sympathetic scrutiny of the nature of the operations, and the scope and the degree of validity of the conclusions, and by interest in their improvement, on the part of modern analysts, to whom minute discussion of the nature and limitations of functional discontinuity in other aspects has become highly congenial. No apology is required for reviewing the practical side of the subject, in an elementary way, before the Mathematical Society, especially as a variation from the purely critical and negative attitude on such matters that is not unfamiliar to us." As a result of this split we were weak in the field of general analysis of the type developed by Poincaré and Birkhoff. It is therefore not surprising to find the more theoretical developments occurring elsewhere. About 1930 Mandelstam and Papalexi in Moscow took up the work of Appleton and van der Pol and, using the work of a French mathematician, Fatou, they supplied a theoretical justification for the approximations just mentioned. Soon afterwards Kryloff and Bogoliuboff in Kiev embarked on a long series of over thirty papers and monographs covering all sorts of problems of non-linear vibrations both electrical and mechanical with a different theoretical justification. They gave an interesting application of some rather highbrow work on differential equations on a torus by another French mathematician, Denjoy. This introduced some subtle ideas in the theory of sets of points. Another Frenchman Liénard introduced a simple topological proof of the existence of a periodic solution of van der Pol's equation suitable for all values of k. This method has, I believe, proved useful in practical applications because it is so easy to generalize.

I should like to sketch briefly a topological treatment of van der Pol's equation which is a modification of Liénard's method. Writing $\dot{x} = y$, we reduce the equation to the first order equation

$$y\frac{dy}{dx} = k(1-x^2)y - x, \quad \text{or} \quad \frac{dy}{dx} = k(1-x^2) - \frac{x}{y}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and we can plot the curves in the x, y plane. If $|x| < 1$, the value of dy/dx shows that the curves cross any circle $x^2 + y^2 = c^2$ outwards as t increases and if $|x| > 1$, the curves cross it inwards. For, since $\dot{x} = y$, x increases with t when $y > 0$ and decreases when $y < 0$. It is easy to see that solutions starting near the origin wind out, and without very much trouble it can be shown that those starting far out wind in. For then the effect of the damping term in $|x| > 1$ outweighs its effect in $|x| < 1$. Bendixon's theory shows that there is a closed curve between the two, and this corresponds to a periodic solution of van der Pol's equation. (Fig. 3).

In 1938 the Department of Scientific and Industrial Research issued a memorandum to the London Mathematical Society appealing for the assistance of pure mathematicians in solving the type of equation occurring in radio work, laying emphasis on the need to know how the frequencies of the periodic solutions varied with the parameters of the equation. This is what attracted my attention and the main reference were to a summary by van der Pol in the *Proceedings of the Institute of Radio Engineers* in 1934 and to a summary by the school of Mandelstam and Papalexi in *Technical Physics of the U.S.S.R.* in 1935. In the summary by van der Pol besides problems in connection with his equation for k large and the synchronization problems mentioned earlier, there were conjectures and statements about a system studied by van der Pol and van der Mark in 1927 which was *supposed* to correspond to van der Pol's equation with forcing term, that is to say, equation (3), with k large. Professor Littlewood and I have had a try at most of these problems, but it was

the observation by van der Pol and van der Mark of two stable solutions with periods $(2n-1)2\pi/\lambda$ and $(2n+1)2\pi/\lambda$ which put us on the track of the most interesting theoretical work. We began in ignorance of the work of Kryloff and Bogoluiboff and, on my part, with little knowledge of the classical work of Poincaré, Liapounov and Birkhoff.

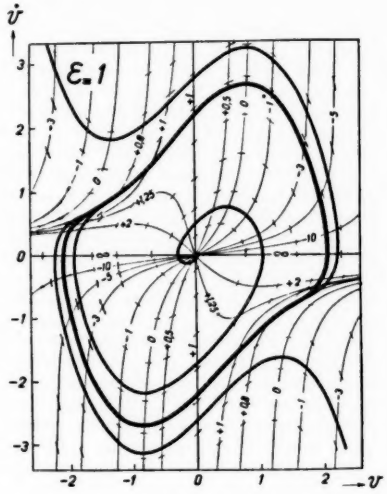

FIG. 3

This is the graphical solution obtained by van der Pol for equation (4) with $x=v$, $k=\epsilon=1$. The thin lines are isoclines with marks to show the slope of the tangent to the solution at various points.

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Meanwhile, the mechanical engineers in the United States, Timoshenko, den Hartog and others had been developing the non-linear theory of springs of variable stiffness and other mechanical problems, and in 1940 von Karman gave a lecture to the American Mathematical Society entitled " The Engineer grapples with non-linear problems ". I am not quite sure whether it was this lecture or problems in connection with naval research on synchronised motors, anti-rolling and automatic steering devices which made Levinson take up the subject. He immediately applied all his skill as an analyst to developing the work of Liénard. With a better training than I had in the classical work of Birkhoff and having read some of the more mathematical work of Kryloff and Bogoliuboff he introduced a topological treatment of equations representing forced oscillations. When Littlewood and I applied this to the equation on which we were working, we found that the experimental results of van der Pol and van der Mark had guided us to an equation with a most remarkable topology suggesting a number of topological problems some of which are still unsolved.

I cannot now remember, but I think that I must have heard of the work of Kryloff and Bogoliuboff before reading Levinson's paper referring to their work. Their most mathematical papers (as distinct from their engineering

papers) are mainly published in Russian although there was a French translation in the library of the Mathematical Association, but it was impossible to find this translation because at that time the library was evacuated and the books were in vertical piles. In the course of my efforts to obtain a copy of this or one of their other papers I wrote to the Society for Cultural Relations with the U.S.S.R., and the secretary of their Science Section told me that over 2000 pages on non-linear vibrations had been published in Russia and that Minorsky was preparing a report on this work for the United States Navy. At her suggestion I wrote to ask for a copy of the report, stating some of the results of Littlewood and myself which had not then been published. I received not only the first part of the report (in spite of the fact that it was classified as Restricted and not to be divulged to unauthorized persons) but also a letter from Lefschetz expressing great interest in our work. Although Lefschetz is famous as a topologist he was trained first as an engineer and could read Russian, and as war work he was organizing research on non-linear vibrations for the United States Navy and in consequence of his work and that of Levinson many pure mathematicians in the United States have taken up the subject, and much of the Russian work has been translated. In 1949 Lefschetz arranged for me to lecture on non-linear differential equations at Princeton and also at Stanford where Minorsky was then working and elsewhere.

There is plenty in the subject to interest a pure mathematician, although perhaps interesting problems of moderate difficulty are getting scarce, but I fear that our assistance to the engineers has not been as great as our own interest in the subject. Some of the difficulties of the radio engineers turned out to be physical rather than mathematical. The systems of the radio engineers have become very complex and the variation between valve and valve is often so great that trial and error is quicker and surer than mathematical calculation. Further, what the pure mathematicians have done shows that the non-linear phenomena are genuinely complicated and no easily applicable general theory can be expected. However, some engineers continue to show a gratifying interest and there is a field scarcely explored from a mathematical point of view in the theory of automatic control systems involved in gun-laying devices and other servo-mechanisms. I am not sufficiently familiar with the theory to explain it but the problem is usually to design an instrument in which no unwanted oscillations will arise and to make the most efficient use of the non-linear element which is an essential part of most control systems. I have not even discovered yet what the typical differential equations are like, for these engineers think in terms of block diagrams, but I have been told that the equations involved are usually of higher order than the second and that the non-linear function is introduced in a fairly simple way. The problem is not only to determine the behaviour of the solutions of the equations, but also to find some way of making its effects easily understood by the practical designer, a problem which should appeal to members of this Association.

M. L. C.

GLEANINGS FAR AND NEAR.

1694. During a lonely night watch Pierre had once calculated that in the twentieth century the mathematical odds against any one human being meeting any other human being were some three hundred times less than would have been the case in the seventeenth century. In the early fifteenth century, before the age of the great navigators, the odds were so vast as to have been a mathematical absurdity.—Robert Standish, *Mr. On Loong*, p. 83. [Per Mr. Sydney Thomson.]

SIGN CONVENTION FOR BENDING MOMENT AND SHEARING FORCE.

By F. T. Chaffer.

If one is concerned merely with the magnitude of the bending moment or the shear, the sign convention to be adopted is of minor importance. When, however, the question of deflection arises, it is essential to obtain the correct sign for the moment and it is therefore desirable to use a convention which not only inspires confidence in obtaining the correct sign but which is also simple to apply.

Among many engineers at the present time the hogging moment is taken as the positive moment. There is much to be said for adopting this convention. It fulfils the conditions mentioned above and provides an easy, and, from the students' point of view, a natural approach to the subject. The following note is merely intended to indicate, somewhat briefly, the method I have adopted for the last few years with students of varying ability.

The simplest type of beam from which one can obtain an insight into the meaning of the terms "bending moment" and "shearing force" is the

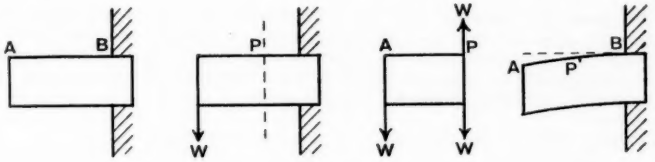

Fig. 1.

cantilever, Fig. 1 (i). If a concentrated load W be placed at A the beam will bend. Consider the portion AP of the beam.

The equilibrium of this portion is unaffected by introducing equal and opposite forces W at P, Fig. 1 (iii); but we can now consider this portion as being in equilibrium under the action of a couple of moment $W \,.\, AP$ called the *bending moment at P on the section AP* and a downward force W at P called the *shearing force at P on the section AP*.

This being the simplest way of loading a beam, it seems reasonable to refer to this type of bending as positive bending. Thus the positive bending moment is the "hogging" moment which tends to deform the beam as shown in Fig. 1 (iv). The essential difference between this approach and the more usual one is that this appeals only to ideas of a mechanical nature and is, so far, independent of the frame of reference. The ease with which the convention can be applied is illustrated by the following examples, which are to be found in most standard texts.

A beam AB of length l is horizontal, simply supported at A and B, and carries a distributed load of uniform intensity w.

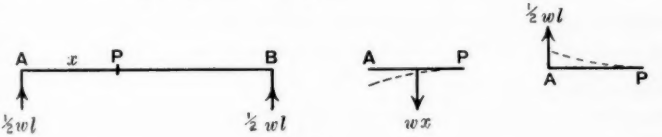

Fig. 2.

By considering the forces acting on AP, we see that :

(i) the load produces a positive bending moment at P of magnitude $\frac{1}{2}wx^2$;

(ii) the reaction produces a negative bending moment of magnitude $\frac{1}{2}wlx$.

Thus, the total bending moment in the positive sense is $\frac{1}{2}wx(x-l)$, that is, the bending moment is everywhere negative.

In the same way the positive shear is $+wx-\frac{1}{2}wl=\frac{1}{2}w(2x-l)$.

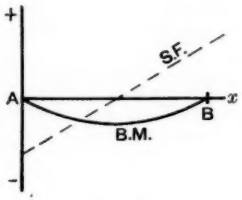

FIG. 3.

If we now take the axis of the beam as the axis of x and a perpendicular to it through A as an axis for measuring shear or bending moment, then the graphs of these quantities are as shown in Fig. 3, and we note that negative shear corresponds to negative slope on the bending moment diagram. This leads to the discussion in the following section.

The relations between loading, shearing force and bending moment.

Consider again the cantilever and isolate a section CD of length δx, where x is the distance of D from the free end B. Then, with the usual notation of

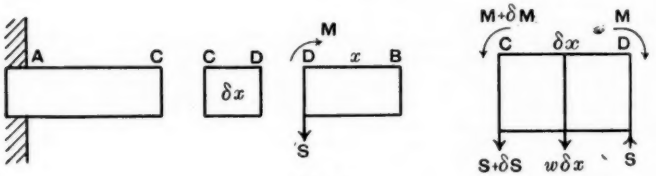

FIG. 4.

the calculus and the above convention of sign, the shear and bending moment at D *on the portion* DB are $S+\delta S$, $M+\delta M$ as shown. Thus, by considering the portion CD, we have, with loading intensity w,

$$S+\delta S=S+w\,.\,\delta x,$$

$$M+\delta M=M+S\,.\,\delta x+w\,.\,\delta x\,.\,\epsilon\,.\,\delta x \quad (0<\epsilon<1),$$

whence $$dS/dx=w. \qquad dM/dx=S,$$

and $$d^2M/dx^2=w.$$

Bending of beams

Consider a cantilever with the end $ABCD$ fixed. Let it bend under its own weight (or an applied load) so that AE becomes stretched to AE', and DH becomes compressed to DH'. There will be some fibre JK whose length

remains unchanged. Let N, R be the centre and radius of curvature of the line JK', and let $JL = z$. Then

$$LM'/JK' = (z + R)/R.$$

Hence, since $JK = JK'$,

$$(LM' - LM)/JK = (LM'/JK') - (LM/JK)$$
$$= z/R, \quad \text{(since } LM = JK\text{)}.$$

Thus the stress intensity in LM' is Ez/R, where E is Young's modulus. Hence the stress in a cylinder whose axis is LM and whose cross-sectional area is δA, is $(Ez/R) \cdot \delta A$. It follows that the total moment of the stress about JQ is

$$\Sigma(Ez^2/R) \cdot \delta A = EI/R,$$

where I is the second moment of area of the section of the beam. But this stress moment is identical with the bending moment. Thus

$$EI/R = M.$$

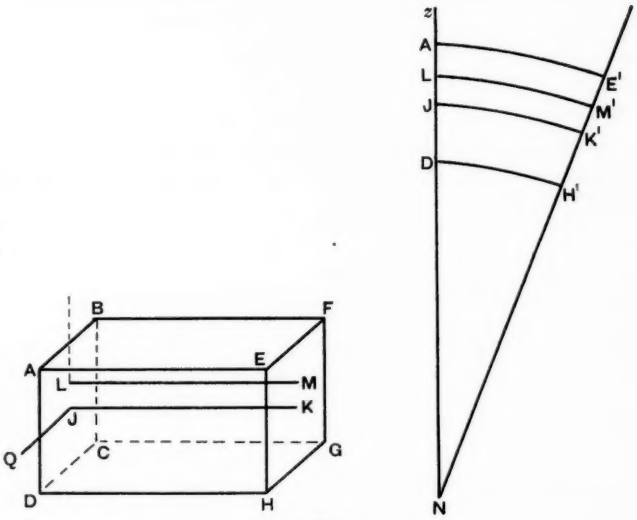

FIG. 5.

If now we take an axis x to lie along the beam in its undeflected position and an axis y in such a direction that its positive sense is in the direction of the applied load, then, with the usual approximation for R the last equation becomes

$$EI \frac{d^2y}{dx^2} = M.$$

In the case of beams which are not horizontal, the " applied load " is to be interpreted as " that force which acting perpendicularly to the beam would cause it to bend in the direction considered ". (Fig. 6).

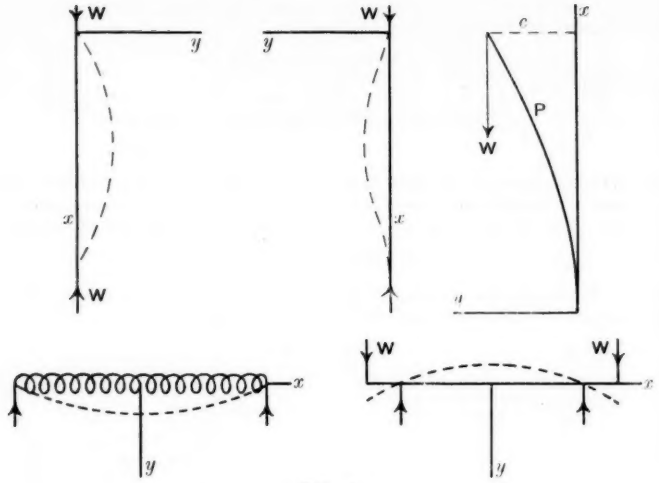

FIG. 6.

The diagrams in Fig. 6 may help to illustrate this point.

The deflection curve for a cantilever of length l carrying a distributed load of uniform intensity w.

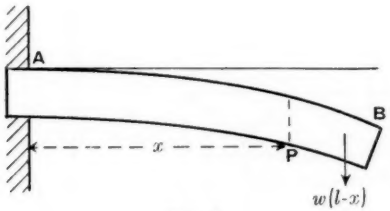

FIG. 7.

The bending moment at P due to the load on PB is $+\frac{1}{2}w(l-x)^2$. Thus

$$EI \,.\, d^2y/dx^2 = \tfrac{1}{2}w(l-x)^2,$$

where $dy/dx = 0$, $y = 0$, when $x = 0$. Thus

$$y = wx^2\{2l^2 + (2l-x)^2\}/24EI,$$

and the beam takes the form shown.

The deflection curve for a beam of length 2a, simply supported in a horizontal position by props at its ends, and carrying a uniformly distributed load of intensity w.

Consider the portion PB, where $OP = x$, $AO = OB$. The bending moment due to the load on PB is $+\frac{1}{2}w(a-x)^2$, while that due to the reaction at B is

$-wa(a-x)$. Hence

$$EI \cdot d^2y/dx^2 = \tfrac{1}{2}w(a-x)^2 - wa(a-x),$$

where $y=0$ when $x=\pm a$.

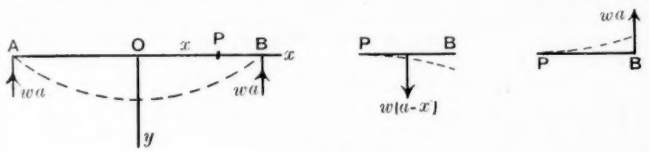

FIG. 8.

Thus

$$y = w(5a^2 - x^2)(a^2 - x^2)/24EI.$$

Hence, y is everywhere positive in the range $-a \leqslant x \leqslant a$ and the deflection curve is that indicated by the dotted line.

A strut is placed in a vertical position and the ends, which are free to move in this vertical line, are subjected to end thrusts W.

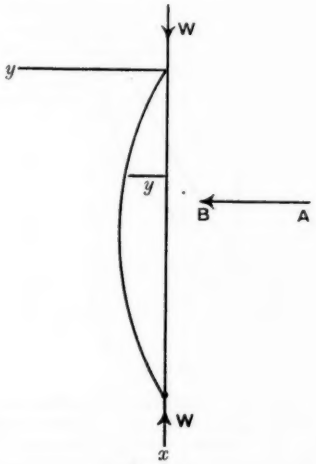

FIG. 9.

The effect of the end thrusts is to make the beam bend as shown (Fig. 9), that is, it behaves as would a simply supported horizontal beam subjected to loading in the direction AB at right angles to the beam. We therefore take the y-axis as shown. The load causes a sagging effect and thus the bending moment at P is $-Wy$. Hence

$$EI \cdot d^2y/dx^2 = -Wy,$$

with a solution of the form

$$y = p \sin nx + q \cos nx, \quad (n^2 = W/EI).$$

A light beam of length l is clamped at its lower end and carries a weight W at the upper end. To determine whether bending or crushing will occur.

Assume that bending occurs and that the top end is deflected horizontally through a distance c (as in Fig. 6 iii). The bending moment at P is $+W(c-y)$, since W is causing a hogging effect. Thus

$$EI \,.\, d^2y/dx^2 = W(c-y)$$

and

$$y = c + p\cos nx + q\sin nx, \quad (n^2 = W/EI)$$

where $y=0$, $dy/dx=0$, when $x=0$, and $y=c$ when $x=l$. This leads quite simply to the condition $\cos ln = 0$, thus showing that the smallest value of l consistent with bending is $\frac{1}{2}\pi\sqrt{(EI/W)}$.

A uniform beam is held inclined to the vertical at an angle α by means of clamps at its ends, and sags under its own weight : to find the fixing moments.

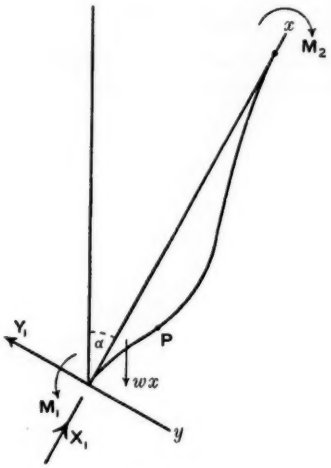

FIG. 10.

Let M_1, X_1, Y_1 be the fixing moment and reaction components at the lower clamp. Let w be the loading intensity and consider a length OP whose unstretched length was x. The shear at P is $+wx\sin\alpha - Y_1$, and the bending moment at P is $EI \,.\, d^2y/dx^2$, with axes chosen as shown in conformity with the above convention. But the shear is dM/dx. Hence

$$EI \,.\, d^2y/dx^2 = wx\sin\alpha - Y_1,$$

and so

$$EI \,.\, d^2y/dx^2 = \tfrac{1}{2}wx^2\sin\alpha - xY_1 + M_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Integrating twice, subject to the conditions $y=0$, $dy/dx=0$ when $x=0$, $x=l$, we obtain

$$M_1 = \tfrac{1}{12}wl^2\sin\alpha, \qquad Y_1 = \tfrac{1}{2}wl\sin\alpha.$$

M_2 is then obtained by putting $x=l$ in equation (i), whence

$$M_2 = \tfrac{1}{12}wl^2\sin\alpha = M_1.$$

F. T. C.

EQUIMOMENTAL SYSTEMS.

By A. Talbot.

1. It is well known that, as regards moment of inertia about any line in its plane, a uniform triangular lamina may be replaced by a set of three particles, each with mass equal to one-third of the mass of the triangle, and placed at the midpoints of the sides. The usual proofs given for this elegant result are rather cumbersome. Now it takes only a few seconds to verify that the set of particles has the same moment of inertia as the triangle about each of the three sides (using the $h^2/6$ formula for the square of the radius of gyration). It is tempting to ask whether the equivalence of particles and lamina may not be deducible from this. In other words, starting from the fact that the equivalence holds for three particular lines in the plane, can it be deduced for *all* lines in the plane?

The work which follows grew out of an investigation of this question.

2. The standard textbooks on Statics or Mechanics say very little about equimomental systems, and what is said usually refers to principal axes and moments, or is confined to laminae. The treatment to be given here will be quite general, and (deliberately) makes no reference to principal axes.

We shall be concerned with continuous or discrete finite systems of positive mass, distributed either along a straight line, or in a plane but not confined to a line, or in space but not confined to a plane. We use the terms *line*, *plane* and *space systems* to denote these mutually exclusive possibilities.

Definition. Two mass-systems are said to be *equimomental* if they have equal second moments (moments of inertia) about any line in space.

Theorem 1. *Two equimomental systems must have the same total mass, and the same centroid.*

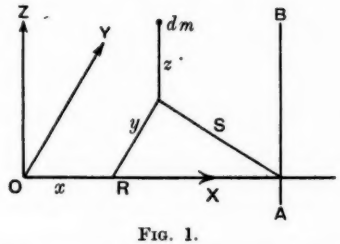

Fig. 1.

Proof. Take any origin O and rectangular axes Ox, Oy, Oz as in Fig. 1. Take a line AB given by $y=0$, $x=R$. A mass dm at (x, y, z), belonging to one of the systems,* is at a distance S from AB, where $S^2=y^2+(R-x)^2$. Thus, if the system has mass M and centroid G (x, y, z), its second moment about AB is

$$I_{AB}=\int S^2\,dm=\int (x^2+y^2)\,dm+R^2\int dm-2R\int x\,dm$$

$$=I_{OZ}+MR^2-2MR\bar{x}.$$

Similarly, using primes for the second system, its moment is

$$I'_{AB}=I'_{OZ}+M'R^2-2M'R\bar{x}'.$$

* For convenience we shall assume throughout the paper that the prescribed systems are continuous. For discrete systems integration must be replaced by summation.

By hypothesis, $I_{OZ} = I'_{OZ}$, and $I_{AB} = I'_{AB}$ for all R. It follows immediately that $M = M'$, and $\bar{x} = \bar{x}'$. Similarly, $\bar{y} = \bar{y}'$, $\bar{z} = \bar{z}'$, so that G and G' coincide.

This theorem shows that equality of all second moments implies equality of mass and of first moments, and so complete equivalence from a dynamical point of view. Thus we may validly use the word "equivalent" instead of "equimomental", and shall use "equivalence" instead of "equimomentality".

It may be noted that this equivalence does *not* follow from equality of second moments about any *finite* number of lines, however large. A proof of this could readily be given if systems with mixed positive and negative masses were included. If, however, we restrict ourselves to positive systems, as we have in fact done, the proof requires the use of results which will be established later, and it is accordingly deferred until section 6.

3. Although we shall discuss space systems in all generality, it is naturally simpler to consider plane and line systems separately. For this purpose the results in the following theorem are needed. They are almost self-evident.

Theorem 2. (*a*) *If two coplanar plane systems have the same second moments about all lines in their plane, they are equimomental.* (*b*) *If two collinear line systems have the same second moments about all points in their line* (*strictly speaking, about all incident perpendiculars*), *they are equimomental.*

To prove (*a*), take the plane as XOY in Fig. 2*a*, the origin O at the intersection of the plane with any line OA in space, making an angle θ with the normal OZ. From any point $(x, y, 0)$ in the plane draw perpendiculars p and p_0 to the line OA and its projection OB respectively. Then (see Fig. 2*a*)

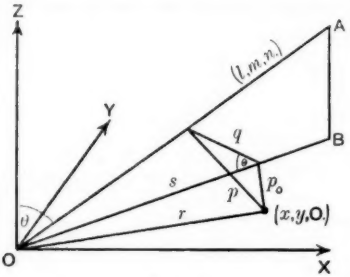

FIG. 2*a*.

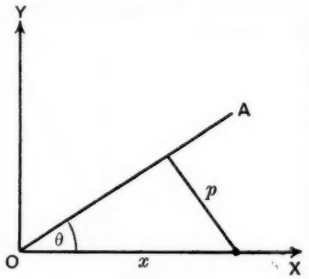

FIG. 2*b*.

$$p^2 = p_0^2 + q^2 = p_0^2 + s^2\cos^2\theta = p_0^2 + (r^2 - p_0^2)\cos^2\theta = p_0^2\sin^2\theta + (x^2 + y^2)\cos^2\theta$$

so that if systems dm, dm' are distributed over the plane, their second moments about OA are

$$I_{OA} = \int p^2\,dm = \sin^2\theta \,.\, I_{OB} + \cos^2\theta \,.\, (I_{OX} + I_{OY})$$

and

$$I'_{OA} = \sin^2\theta \,.\, I'_{OB} + \cos^2\theta \,.\, (I'_{OX} + I'_{OY})$$

and by hypothesis these are equal term by term.

To prove (*b*), let OX be the line (Fig. 2*b*), with O any point on it, and let OY be any perpendicular through O. If OA is any line through O in the plane XOY, and systems dm, dm' are distributed over OX, we have $p = x\sin\theta$, so that

$$I_{OA} = \sin^2\theta \cdot I_{OY}, \qquad I'_{OA} = \sin^2\theta \cdot I'_{OY},$$

and these are equal since $I_{OY} = I'_{OY}$.

Thus the systems are equimomental as regards the plane XOY, and hence totally so, by (a).

4. We are now ready to tackle the two problems considered in this paper, both of them suggested by the triangle result mentioned earlier. They are :

(i) to find the smallest number of lines about which equality of second moments of two systems ensures equivalence, and

(ii) to find the smallest number of particles that can form a system equivalent to a given mass system, and to find such a minimum equivalent particle system.

The answer to problem (i) is fairly obvious, and is contained in the following three theorems. Problem (ii) is solved in section 5.

Theorem **3.** *If two space systems have the same mass and centroid, and equal second moments about any* 6 *lines (not parallel to the generators of a single quadric cone), they are equimomental ; and fewer lines do not suffice to ensure this.*

That the identity of mass and centroid must be explicitly mentioned follows from the remarks at the end of section 2. Let the common mass be M and the common centroid G. It then follows from the Parallel Axes Theorem that if the systems have equal second moments about any line, the same holds for a parallel line through G, or any other parallel line.

We first prove a purely geometrical result. Suppose that we have any N lines, denoted by L_i ($i = 1, 2, \ldots, N$), through a point G. Taking rectangular axes with G as origin, let the lines have direction-cosines (l_i, m_i, n_i), and let $P(x, y, z)$ be any point in space.

If p_i is the perpendicular distance from P to L_i, we have

$$p_i^2 = (yn_i - zm_i)^2 + (zl_i - xn_i)^2 + (xm_i - yl_i)^2, \quad \ldots\ldots\ldots\ldots(1)$$

so that if u_i are arbitrary multipliers,

$$\sum_{i=1}^{N} u_i p_i^2 = x^2 \Sigma u_i (m_i^2 + n_i^2) + \; + \; - 2yz \Sigma u_i m_i n_i - \; - \quad \ldots\ldots\ldots\ldots(2)$$

This vanishes identically in x, y and z, *i.e.* for all P, if and only if

$$\sum_{1}^{N} u_i l_i^2 = \Sigma u_i m_i^2 = \Sigma u_i n_i^2 = \Sigma u_i m_i n_i = \Sigma u_i n_i l_i = \Sigma u_i l_i m_i = 0. \quad \ldots\ldots\ldots(3)$$

These are 6 homogeneous equations in the N variables u_i. In general there is a solution not identically zero if and only if $N \geqslant 7$. In particular, if $N = 7$, none of the u_i vanishes, and any one p_i^2, say p_7^2, is a linear combination of the other six, these being linearly independent.

In other words, we have the geometrical result that in general, given any six fixed lines L_i ($i = 1, \ldots, 6$) and an arbitrary point P, the distance-squares p_i^2 ($i = 1, \ldots, 6$) are linearly independent, but the distance-square p_7^2 to any seventh line is linearly dependent on the first six p_i^2, say

$$p_7^2 = \sum_{i=1}^{6} v_i p_i^2, \quad \ldots\ldots\ldots\ldots(4)$$

the coefficients v_i being of course independent of the position of P, and determined only by the directions of the lines L_i ($i = 1, \ldots, 7$).

Now, if the two mass systems concerned have equal second moments I_i, I_i' about six lines L_i ($i = 1, \ldots, 6$), through G, let L_7 be any other line through G.

Then

$$I_7 = \int p_7^2 dm = \sum_1^6 v_i I_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

using (4), and

$$I_7' = \sum_1^6 v_i I_i'.$$

Since $I_i = I_i'$ $(i = 1, \ldots, 6)$, it follows that $I_7 = I_7'$, *i.e.* the systems have equal second moments about any other line.

An exceptional case would occur if equations (3) had a solution with $N = 6$. Then the determinant of the coefficients l_i^2, etc., would vanish, and the rows l_i^2, etc., would be linearly dependent, *i.e.* connected by a relation of the form

$$al_i^2 + bm_i^2 + cn_i^2 + dm_in_i + en_il_i + fl_im_i = 0 \quad (i = 1, \ldots, 6).$$

The lines would then, however, all be generators of the quadric cone

$$ax^2 + by^2 + cz^2 + dyz + ezx + fxy = 0,$$

which case is explicitly excluded by hypothesis.

Thus we have proved the first part of the theorem.

If the special case does occur, then at least one of the I_i, say I_6, is completely defined by the other five, any seventh I_7 (for a line L_7 not on the quadric cone uniquely determined by $L_1, \ldots, L_5$) is arbitrary, and any L_8 is determined by $I_1, \ldots, I_5, I_7$.

The second part of the theorem—that equality of moments about 5 lines does not suffice to ensure equivalence—is most easily proved with the help of results in the next section, and the proof is deferred until section 6.

Theorem 4. *If two coplanar plane systems have the same mass and centroid, and equal second moments about any* 3 *lines in their plane, they are equimomental; and fewer lines do not suffice to ensure this.*

By Theorem 2a we may confine our attention to the plane of the systems. The proof of the second part will be deferred until section 6. For the first part we give two different proofs.

The first proof is similar to that of Th. 3. Take rectangular axes through the centroid G, and any N lines L_i $(i = 1, \ldots, N)$, through G, in the plane, with direction cosines (l_i, m_i). If (x, y) is any point in the plane, and p_i its distances from the lines L_i,

$$p_i^2 = (xm_i - yl_i)^2,$$

so that

$$\sum_{i=1}^N u_i p_i^2 = x^2 \Sigma u_i m_i^2 + y^2 \Sigma u_i l_i^2 - 2xy \Sigma u_i l_i m_i.$$

This vanishes for all P if and only if

$$\sum_1^N u_i l_i^2 = \Sigma u_i m_i^2 = \Sigma u_i l_i m_i = 0.$$

These three equations in the N variables u_i have a non-zero solution if, and in general only if, $N \geqslant 4$. Moreover, the exceptional case in which there is a solution for $N = 3$ is excluded, for it would imply (cf. the proof of Th. 3) that l_i^2, m_i^2, l_im_i satisfy a relation of the form

$$al_i^2 + bm_i^2 + fl_im_i = 0 \qquad (i = 1, 2, 3).$$

Thus, l_i/m_i would have only *two* possible values, and two of the three lines L_i would coincide.

Taking $N = 4$, it follows as before that any four distance-squares are linearly dependent, while any three are independent, so that if any three fixed lines L_1, L_2, L_3 are given, and any fourth line chosen, then for any point P,

$$p_4^2 = \sum_{i=1}^{3} v_i p_i^2, \quad \ldots\ldots(6)$$

where the v_i depend on the directions of the four lines, but not on the position of P.

It then follows that if two systems have equal moments about three lines L_i $(i = 1, 2, 3)$ through G, and L_4 is any other line,

$$I_4 = \sum_{1}^{3} v_i I_i, \qquad I_4' = \sum_{1}^{3} v_i I_i',$$

and these are equal since

$$I_i = I_i' \quad (i = 1, 2, 3).$$

The second proof is based on the fact that the three given lines may be taken as reference lines for a set of trilinear coordinates. In fact, if p, q, r are the distances of any point P from the sides a, b, c of any triangle, of area Δ, we have

$$ap + bq + cr = 2\Delta. \quad \ldots\ldots(7)$$

If, in particular, the triangle is shrunk by parallel movements of its sides to an internal point G,

$$ap + bq + cr = 0, \quad \ldots\ldots(8)$$

where now p, q, r are the distances of P from any three lines L_a, L_b, L_c through G, and a, b, c are non-zero coefficients proportional to the sines of the angles between the lines (as in the original triangle). As regards signs, it is easy to see that if a, b, c are taken as positive, then we may take p, q, r as positive if

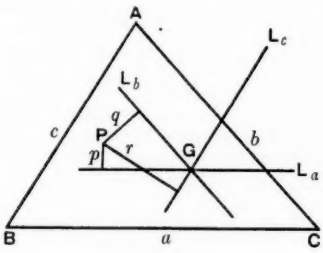

FIG. 3.

measured on the same side of L_a, L_b, L_c as the vertices A, B, C respectively of any triangle of parallel sides a, b, c enclosing G. For example, in Fig. 3, p and q are positive, r is negative. (They cannot all be simultaneously positive.) G, of course, is not (necessarily) the centroid of $\triangle ABC$, but will be taken as the centroid of the mass systems to be discussed.

Suppose now two systems of equal mass and common centroid G have equal second moments about three lines L_a, L_b, L_c. If p, q, r, s are the distances of an arbitrary point P from L_a, L_b, L_c and any fourth line L, then we may write, by (8),

$$-ap = bq + cr,$$

and similarly

$$-as = dq + er.$$

Squaring and combining to eliminate qr, we obtain a relation of the form

$$s^2 = up^2 + vq^2 + wr^2,$$

analogous to (6). The proof then proceeds as before.

This second proof has been given since we shall use the (p, q, r) coordinate system in section 5, when discussing the construction of plane equimomental systems.

Theorem 5. If two collinear line systems have the same mass and centroid and equal second moments about one point in their line, they are equimomental.

This follows immediately from the Parallel Axes Theorem, for if A and B are any two points in the line, M the common mass and G the common centroid,

$$I_B = I_G + M \,.\, GB^2 = I_A + M \,.\, (GB^2 - GA^2),$$

$$I_B' = I_A' + M \,.\, (GB^2 - GA^2),$$

whence $I_B = I_B'$ if $I_A = I_A'$.

5. The preceding theorems simplify the task of determining whether two given systems are equimomental. For example, the question relating to a triangular lamina raised in section 1 can be answered in the affirmative in view of Theorem 4. A problem of equal interest, however, is to determine whether, if a mass-system is given, there is an equivalent particle system with a prescribed number of positive particles, and to find the minimum possible number of particles. A complete solution of this problem is given here. We shall consider line, plane and space systems successively.

Line Systems. Let a given line system have mass M, centroid G, and second moment Mk^2 about G. Take G as origin for coordinates.

Any equimomental system of n particles m_s at x_s must, by Theorems 1 and 5, satisfy three necessary and sufficient conditions :

$$\sum_{s=1}^{n} m_s = M,$$

$$\Sigma m_s x_s = 0,$$

$$\Sigma m_s x_s^2 = Mk^2.$$

It is clear that one particle alone cannot satisfy these. We show that two particles do suffice, and there is an infinite choice (with one degree of freedom). With two particles, the conditions become

$$\left.\begin{array}{ll} \text{(i)} & m_1 + m_2 = M, \\ \text{(ii)} & m_1 x_1 + m_2 x_2 = 0, \\ \text{(iii)} & m_1 x_1^2 + m_2 x_2^2 = Mk^2. \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

By multiplying (i) and (iii), and subtracting the square of (ii), it follows that

$$m_1 m_2 (x_1 - x_2)^2 = M^2 k^2,$$

whence

$$x_1/m_1 = -x_2/m_2 = (x_1 - x_2)/M = k/\sqrt{(m_1 m_2)}.$$

Thus, if we take

$$m_1 = \alpha M, \qquad m_2 = (1 - \alpha) M, \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where α is arbitrary in the range $0 < \alpha < 1$, then

$$x_1 = k\sqrt{\{(1 - \alpha)/\alpha\}}, \qquad x_2 = -k\sqrt{\{\alpha/(1 - \alpha)\}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

the square roots being taken with the same sign, either + or −.

In particular, taking $\alpha = 1/2$, we obtain the equal-particle solution

$$m_1 = m_2 = M/2, \qquad x_1 = -x_2 = k. \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Alternatively, we may prescribe the coordinate x_1, say, as βk, where β is entirely arbitrary. Then $\alpha = 1/(1+\beta^2)$, $m_1 = M/(1+\beta^2)$, $m_2 = \beta^2 M/(1+\beta^2)$, $x_1 = -k/\beta$.

Plane Systems. Let a given plane system have mass M, centroid G, and second moments $M\kappa^2$, $M\lambda^2$, $M\mu^2$ about three lines L_a, L_b, L_c through G. Take coordinates p, q, r as in Fig. 3. These are always connected by a relation

$$ap + bq + cr = 0. \quad \text{.....(13)}$$

Any equimomental system of n particles m_s at (p_s, q_s, r_s) must, by Theorems 1 and 4, satisfy the necessary and sufficient conditions

$$\left.\begin{aligned} &\text{(i)} \ \sum_{s=1}^{n} m_s = M \\ &\text{(ii)} \ \Sigma m_s p_s = 0 \\ &\text{(iii)} \ \Sigma m_s q_s = 0 \\ &\text{(iv)} \ \Sigma m_s r_s = 0 \\ &\text{(v)} \ \Sigma m_s p_s^2 = M\kappa^2 \\ &\text{(vi)} \ \Sigma m_s q_s^2 = M\lambda^2 \\ &\text{(vii)} \ \Sigma m_s r_s^2 = M\mu^2 \end{aligned}\right\} \quad \text{.....(14)}$$

while the coordinates must satisfy

$$ap_s + bq_s + cr_s = 0 \quad (s = 1, 2, \ldots, n). \quad \text{.....(15)}$$

In view of (15), (14, iv) is an automatic consequence of (ii) and (iii). (There can of course be only two first-moment conditions for a plane system.) We thus have, again in view of (15), 6 conditions for the $3n$ variables m_s, p_s, q_s. Thus n must be at least 2. But two particles cannot be equivalent to a plane (non-line) system. For they have zero second moment about their join, while that of the system must be positive. This impossibility could also be shown algebraically, but we omit the proof here.

We must thus take $n \geqslant 3$. In fact it will now be shown that we may take $n = 3$. This is clearly sufficient for algebraically solving the equations (there being $9 - 6 = 3$ degrees of freedom). What we have to show is that the solutions include *real positive-particle* systems.

Our equations are now (14) and (15) with $n = 3$. Consider first the pair (14) (ii) and (v). By means of (ii) we can express p_3 in terms of p_1 and p_2, and so $\sum_{s=1}^{3} m_s p_s^2$ as a quadratic form in p_1 and p_2. The simplest procedure then is to express this as a sum of squares, in any manner. For example, we have

$$\begin{aligned} m_3 \sum_{1}^{3} m_s p_s^2 &= m_3 m_1 p_1^2 + m_2 m_3 p_2^2 + (m_1 p_1 + m_2 p_2)^2 \\ &= m_2(m_2 + m_3)\left(p_2 + \frac{m_1}{m_2 + m_3} p_1\right)^2 + \frac{M m_1 m_3}{m_2 + m_3} p_1^2, \end{aligned}$$

using (i). Treating the other equations likewise, we find we may write them as

$$P_1^2 + \alpha P^2 = \beta\kappa^2, \quad \text{.....(16)}$$

$$Q_1^2 + \alpha Q^2 = \beta\lambda^2, \quad \text{.....(17)}$$

$$R_1^2 + \alpha R^2 = \beta\mu^2, \quad \text{.....(18)}$$

where

$$\alpha = m_1 m_3 / M m_2, \quad \beta = m_1/(m_2 + m_3), \quad P_1 = \beta p_1, \quad P = p_2 + \beta p_1,$$

and similarly for Q_1, Q, R_1, R.

Now, by (15),

$$cR = -(aP + bQ), \qquad cR_1 = -(aP_1 + bQ_1).$$

Substituting in (18) and using (16) and (17), we obtain

$$P_1Q_1 + \alpha PQ = \beta\rho \quad \ldots\ldots(19)$$

where

$$\rho = (c^2\mu^2 - a^2\kappa^2 - b^2\lambda^2)/2ab. \quad \ldots\ldots(20)$$

The solution now proceeds straightforwardly from (16), (17) and (19), in which P, P_1, Q and Q_1 are independent variables. We may eliminate P and Q and obtain

$$\alpha^2P^2Q^2 = (\beta\kappa^2 - P_1^2)(\beta\lambda^2 - Q_1^2) = (\beta\rho - P_1Q_1)^2, \quad \ldots\ldots(21)$$

which on expansion and rearrangement gives

$$(\kappa^2Q_1 - \rho P_1)^2 = (\beta\kappa^2 - P_1^2)(\kappa^2\lambda^2 - \rho^2). \quad \ldots\ldots(22)$$

It is thus clear that for any choice of positive m_1, m_2, m_3 with sum M, a real solution is obtained on taking arbitrary p_1 making $p_1^2 \leqslant \kappa^2/\beta$, *i.e.* $P_1^2 \leqslant \beta\kappa^2$, and so $Q_1^2 \leqslant \beta\lambda^2$, by (21), provided the necessary and sufficient condition

$$\kappa^2\lambda^2 \geqslant \rho^2$$

is satisfied. That this does indeed hold can be seen as follows.

For the given system we have, by definition,

$$\int p^2\,dm = M\kappa^2, \qquad \int q^2\,dm = M\lambda^2, \qquad \int r^2\,dm = M\mu^2,$$

whence by (13)

$$Mc^2\mu^2 = \int (ap + bq)^2\,dm = Ma^2\kappa^2 + Mb^2\lambda^2 + 2ab\int pq\,dm,$$

or by (20)

$$\int pq\,dm = M\rho.$$

Thus, for any u and v,

$$\int (up + vq)^2\,dm = M(\kappa^2u^2 + 2\rho uv + \lambda^2v^2).$$

The left-hand side is positive for all u and v (since we are dealing with positive mass systems), whence $\rho^2 - \kappa^2\lambda^2 < 0$, as required. (This is really an example of Schwarz's Inequality for integrals.)

Our solution may then be formulated as follows. Given any plane system of mass M, take any three convenient lines through the centroid as reference lines, and let the sines of the angles between them be numerically equal to or proportional to a, b, c. Let the system's second moments about the lines be $M\kappa^2$, $M\lambda^2$, $M\mu^2$. Calculate ρ by (20), and $\sigma = \sqrt{\{\alpha(\kappa^2\lambda^2 - \rho^2)\}}$.

Choose any positive m_1, m_2, m_3 with sum equal to M, and calculate

$$\alpha = m_1m_2/Mm_3, \qquad \beta = m_1/(m_2 + m_3).$$

Take any p_1 numerically not exceeding $\kappa/\sqrt{\beta}$. Evaluate in succession $P_1 = \beta p_1$,

$$P = \sqrt{\{(\beta\kappa^2 - P_1^2)/\alpha\}}, \qquad q_1 = (\rho P_1 + \sigma P)/\beta\kappa^2,$$

$$Q = \beta(\rho - P_1q_1)/\alpha P = (\alpha\rho P - \sigma P_1)/\alpha\kappa^2, \qquad p_2 = P - P_1,$$

$$p_3 = -(m_1p_1 + m_2p_2)/m_3 = -(P_1 + m_2P/m_3), \qquad q_2 = Q - \beta q_1,$$

$$q_3 = -(\beta q_1 + m_2Q/m_3).$$

Then (p_s, q_s) determine the positions of three masses m_s which form a system equivalent to the given one. (The r_s may be found from (15) if required.)

The square-root signs for σ and P may both be chosen arbitrarily + or −.

It is clear that if (p_s, q_s) is any one solution, using values P and σ, so is $(-p_s, -q_s)$, obtained by starting with $-p_1$ and using $-P$ and σ. That $(-p_s, -q_s)$ is also a solution is of course immediately obvious from (14) and (15), the original equations. We may call it the "image" of the solution (p_s, q_s).

A special limiting case of our solution occurs if we give p_1 its maximum permitted numerical value, $\kappa/\sqrt{\beta}$. Then we find

$$P=0, \qquad q_1=\rho/\beta p_1, \qquad Q=-\sigma/\alpha p_1,$$

$$p_2=-\beta p_1=p_3, \qquad q_2=-\left(\rho+\frac{\sigma}{\alpha}\right)\Big/p_1, \qquad q_3=\left(-\rho+\frac{m_2}{m_3}\frac{\sigma}{\alpha}\right)\Big/p_1.$$

A solution of particular interest is that in which the three masses m_1, m_2, m_3 are equal, each being $M/3$. Then we have $\alpha=1/3$, $\beta=1/2$.

Many variations of the above solution are possible, For example, instead of prescribing the masses, and p_1, one could prescribe p_1, p_2, p_3. Then we should find the masses $m_1=(\kappa^2+p_2p_3)M/(p_1-p_2)(p_1-p_3)$, etc., and then the q's as before. For this to be a positive solution, if we suppose $p_1<p_2<p_3$, we must have $p_1<0$, $p_3>0$, $|p_1p_3|>\kappa^2$, and $|p_2p_3|$ or $|p_1p_2|<\kappa^2$ according as $p_2<0$ or >0.

An analogous solution to the above one could of course be set down in terms of rectangular coordinates. We omit this here since it will presently be done for the more complicated case of space systems.

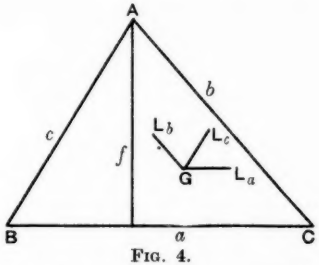

FIG. 4.

As an example of the whole preceding process, consider a triangular lamina as shown in Fig. 4. Let a, b, c be the sides, G the centroid, f, g, h the perpendiculars from the vertices to the opposite sides (only f is shown). Then if Δ is the area of the lamina, which we suppose of unit density, so that $M=\Delta$, we have

$$af=bg=ch=2\Delta,$$

$$I_{BC}=\Delta f^2/6, \quad I_{CA}=\Delta g^2/6, \quad I_{AB}=\Delta h^2/6.$$

Taking L_a, L_b, L_c through G parallel to the sides, we may take a, b, c as the coefficients in (13). Then $I_{L_a}=\Delta f^2/6-\Delta(f/3)^2=\Delta f^2/18$, by the Parallel Axes Theorem. Thus, $\kappa^2=f^2/18$, and similarly $\lambda^2=g^2/18$, $\mu^2=h^2/18$, so that

$$\rho=(c^2h^2-a^2f^2-b^2g^2)/36ab=-fg/36,$$

and

$$\sigma=\sqrt{(3\alpha)}fg/36.$$

Of all possible solutions, let us now restrict ourselves to equal-particle solutions, and take in particular the limiting case $p_1=-\kappa/\sqrt{\beta}=-f/3$ (using the $-$ sign so as to make m_1 lie on BC). Then $q_1=g/6$, $p_2=p_3=f/6$, $q_2=g/6$, $q_3=-g/3$. This gives the familiar equimomental system of particles $M/3$ at

the midpoints of the sides. The image system has particles at the midpoints of AG, BG and CG.

Space Systems. Let a given space system have mass M, centroid G, and take rectangular axes at G. Any equimomental system of n particles m_s at (x_s, y_s, z_s) must, by Theorems 1 and 3, satisfy 10 conditions in all (equal mass, 3 equal first moments, 6 equal second moments). Since there are $4n$ variables, m_s, x_s, y_s, z_s, n must be at least 3. But n cannot be 3, for then the 3 particles would form a plane system, which cannot be equimomental with a space system. (This is intuitively obvious. It may be proved by using the second part of Theorem 3. For a plane system's second moments are all determined by those about 3 lines, whereas by Theorem 3, 6 lines are needed for a space system.)

We must therefore take $n \geqslant 4$. We shall now show that real positive systems of 4 particles can be constructed to be equivalent to any given space system. There are then $16 - 10 = 6$ degrees of freedom.

Now take rectangular axes at G, any six lines L_i (l_i, m_i, n_i), $(i = 1, \ldots, 6)$, and particles m_s at $P_s(x_s, y_s, z_s)$, $(s = 1, \ldots, 4)$. If the given system has second moments I_i about L_i,

$$I_i = \int p_i^2 dm$$

$$= (m_i^2 + n_i^2) \int x^2 dm + \ldots + \ldots - 2m_i n_i \int yz\, dm - \ldots - \ldots, \quad (i = 1, \ldots 6). \quad \ldots(23)$$

The conditions for the particle system are

$$\sum_{s=1}^{4} m_s = M,$$

$$\Sigma m_s x_s = 0,$$

$$\Sigma m_s y_s = 0,$$

$$\Sigma m_s z_s = 0,$$

$$\Sigma m_s p^2_{is} = I_i, \quad (i = 1, \ldots, 6),$$

where p_{is} is the distance from P_s to L_i, and

$$p^2_{is} = (m_i^2 + n_i^2) x_s^2 + + - 2m_i n_i y_s z_s - - .$$

By comparison with (23) it is obvious that the second moment conditions imply (if the 6 lines are not common generators of a quadric cone)

$$\sum_{s=1}^{4} m_s x_s^2 = \int x^2 dm, \text{ etc.},$$

and

$$\sum_{s=1}^{4} m_s y_s z_s = \int yz\, dm, \text{ etc.}$$

(These, of course, make the two systems have the same momental ellipsoid, *i.e.* the same moments and products of inertia with respect to the axes. Thus they could have been written down at once. We have preferred to reach them via the moments about six arbitrary lines. These lines will however be disregarded henceforth.)

If we use the notations

$$\left.\begin{array}{lll} \int x^2 dm = Mg_1, & \int y^2 dm = Mg_2, & \int z^2 dm = Mg_3, \\ \int yz\, dm = Mh_1, & \int zx\, dm = Mh_2, & \int xy\, dm = Mh_3, \end{array}\right\} \quad \ldots\ldots\ldots(24)$$

we may write our equations

$$\sum_{s=1}^{4} m_s = M, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

$$\sum_{1}^{4} m_s x_s = 0, \qquad \Sigma m_s y_s = 0, \qquad \Sigma m_s z_s = 0, \quad \ldots\ldots\ldots\ldots(26)$$

$$\sum_{1}^{4} m_s x_s^2 = Mg_1, \qquad \Sigma m_s y_s^2 = Mg_2, \qquad \Sigma m_s z_s^2 = Mg_3, \ldots\ldots\ldots(27)$$

$$\sum_{1}^{4} m_s y_s z_s = Mh_1, \qquad \Sigma m_s z_s x_s = Mh_2, \qquad \Sigma m_s x_s y_s = Mh_3 \ldots\ldots\ldots(28)$$

Using (26, i) with (27, i) we may eliminate x_4 from $\sum_{1}^{4} m_s x_s^2$, and then express this quadratic form in a standard way as a sum of squares (just as we did for plane systems). We have, for example, making use of (25) also,

$$m_4 \sum_{1}^{4} m_s x_s^2 = m_4(m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^2) + (m_1 x_1 + m_2 x_2 + m_3 x_3)^2$$

$$= m_3(m_3 + m_4)\left(\frac{m_1}{m_3 + m_4} x_1 + \frac{m_2}{m_3 + m_4} x_2 + x_3\right)^2$$

$$+ \frac{m_3 m_4 (M - m_1)}{m_3 + m_4}\left(x_2 + \frac{m_1}{M - m_1} x_1\right)^2 + m_4 M \frac{m_1}{M - m_1} x_1^2.$$

Thus, if we write

$$c^2 = \frac{m_1}{M - m_1}, \qquad d^2 = \frac{m_3 m_4}{m_2(M - m_1)}, \qquad e = \frac{M(m_3 + m_4)}{m_3 m_4}, \quad \ldots\ldots\ldots(29)$$

and

$$X = cx_1, \qquad X' = (x_2 + c^2 x_1)/d, \qquad X'' = \left(1 + \frac{m_3}{m_4}\right) x_3 + \frac{m_2}{m_4} x_2 + \frac{m_1}{m_4} x_1,$$

with similar expressions for Y and Z, etc., we may re-write (27) as

$$\left.\begin{aligned} X'^2 + X''^2 &= e(g_1 - X^2) = ea_1, \quad \text{say,} \\ Y'^2 + Y''^2 &= e(g_2 - Y^2) = ea_2, \\ Z'^2 + Z''^2 &= e(g_3 - Z^2) = ea_3. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

In the same way we find, either by direct evaluation or by applying the theory of quadratic and bilinear forms under linear transformations, that (28) may be re-written

$$\left.\begin{aligned} Y'Z' + Y''Z'' &= e(h_1 - YZ) = eb_1, \quad \text{say,} \\ Z'X' + Z''X'' &= e(h_2 - ZX) = eb_2, \\ X'Y' + X''Y'' &= e(h_3 - XY) = eb_3. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(31)$$

In (30) and (31) we have put terms in X, Y, Z on the right-hand sides since we shall (temporarily) regard them as parameters for our solution.

The simplest method of dealing with (30) and (31) is to use trigonometrical substitutions. Let us write

$$\left.\begin{aligned} X' &= \surd(ea_1) \,.\, \cos\theta_1, & X'' &= \surd(ea_1) \,.\, \sin\theta_1, \\ Y' &= \surd(ea_2) \,.\, \cos\theta_2, & Y'' &= \surd(ea_2) \,.\, \sin\theta_2, \\ Z' &= \surd(ea_3) \,.\, \cos\theta_3, & Z'' &= \surd(ea_3) \,.\, \sin\theta_3. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

Then (30) are automatically satisfied, and (31) become

$$\left.\begin{aligned} \cos(\theta_2 - \theta_3) &= b_1/\surd(a_2 a_3), \\ \cos(\theta_3 - \theta_1) &= b_2/\surd(a_3 a_1), \\ \cos(\theta_1 - \theta_2) &= b_3/\surd(a_1 a_2). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(33)$$

It is now clear that if the masses m_s are chosen first, the quantities X, Y, Z cannot be chosen arbitrarily, for since the sum of the angles $(\theta_2 - \theta_3)$, etc., is zero, the ratios $b_1/\sqrt{(a_2a_3)}$, etc., must be functionally connected. The connection involved may be expressed in many ways. For example, since

$$\cos(\theta_2 - \theta_1) = \cos[(\theta_2 - \theta_3) + (\theta_3 - \theta_1)],$$

$$\frac{b_3}{\sqrt{(a_1a_2)}} = \frac{b_1}{\sqrt{(a_2a_3)}} \cdot \frac{b_3}{\sqrt{(a_3a_1)}} - \frac{\sqrt{(a_2a_3 - b_1^2)}}{\sqrt{(a_2a_3)}} \cdot \frac{\sqrt{(a_3a_1 - b_3^2)}}{\sqrt{(a_3a_1)}},$$

i.e. $$(b_1b_2 - a_3b_3)^2 = (a_2a_3 - b_1^2)(a_3a_1 - b_3^2),$$

or $$B_3^2 = A_1A_2,$$

where A_1, A_2, B_3 are the cofactors of a_1, a_2, b_3 in the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{vmatrix}.$$

This must therefore vanish, since $A_1A_2 - B_3^2$, being the minor of A_3 in the determinant formed from cofactors of elements of Δ, is equal to the product $a_3\Delta$, by a well-known general property of determinants, which we shall use again presently.

Thus the required condition on X, Y, Z is that

$$\Delta = 0.$$

We now show that for any choice of m_s satisfying (25), it is possible to choose X, Y, Z so as to satisfy $\Delta = 0$ and to produce a real solution of the problem. The conditions for this are as follows: X, Y, Z are real; $a_1, a_2, a_3 \geqslant 0$; $A_1, A_2, A_3 \geqslant 0$ (which ensure that $\cos^2(\theta_2 - \theta_3) \leqslant 1$, etc.).

Now it is easy to show that all conditions except possibly the reality of X can be satisfied by choosing real Y and Z such as to make $A_1 \equiv a_2a_3 - b_1^2 \geqslant 0$, and then X to satisfy $\Delta = 0$.

To see this, let

$$D \equiv \begin{vmatrix} g_1 & h_3 & h_2 \\ h_3 & g_2 & h_1 \\ h_2 & h_1 & g_3 \end{vmatrix}, \qquad \text{.....(34)}$$

and let G_r be the cofactors of g_r in D, H_r the cofactors of h_r.

D is the determinant of the quadratic form

$$g_1u^2 + g_2v^2 + g_3w^2 + 2h_1vw + 2h_2wu + 2h_3uv,$$

which is, by (24), equal to

$$\frac{1}{M}\int (ux + vy + wz)^2 dm,$$

and is thus positive-definite, for a positive mass non-plane space system. Thus D, G_1, G_2, G_3 are all positive, in addition to g_1, g_2, g_3.

Now

$$\begin{aligned} A_1 = a_2a_3 - b_1^2 &= (g_2 - Y^2)(g_3 - Z^2) - (h_1 - YZ)^2 \\ &= G_1 + 2h_1YZ - g_3Y^2 - g_2Z^2 \\ &= G_1 - [(g_2Z - h_1Y)^2 + G_1Y^2]/g_2. \end{aligned}$$

Thus, if $A_1 \geqslant 0$, it follows that $Y^2 \leqslant g_2$, *i.e.* $a_2 \geqslant 0$, and similarly $a_3 \geqslant 0$. Moreover, since $A_1A_2 = B_3^2$, $A_2 \geqslant 0$, and similarly, since $A_1A_3 - B_2^2 = a_2\Delta = 0$, $A_3 \geqslant 0$. Finally, since $A_3 = a_1a_2 - b_3^2$, we must have $a_1 \geqslant 0$. This proves the statement above. Thus there only remains the question of the reality of X, obtained by solving $\Delta = 0$ for X, when Y and Z are given certain prescribed

values. Now, each element of Δ is a difference of two terms, and so Δ may be expressed as the sum of 8 determinants, of which 4 are seen to vanish through having two or three columns proportional. Thus we obtain

$$\Delta = D - \begin{vmatrix} X^2 & h_3 & h_2 \\ XY & g_3 & h_1 \\ XZ & h_1 & g_3 \end{vmatrix} - \text{two similar determinants},$$

whence we find

$$-\Delta = G_1X^2 + 2(H_2Z + H_3Y)X + (G_2Y^2 + 2H_1YZ + G_3Z^2 - D).$$

The roots of $\Delta = 0$ in X are real provided the discriminant is positive. This is

$$\begin{aligned}(H_2Z + H_3Y)^2 - G_1(G_2Y^2 + 2H_1YZ + G_3Z^2 - D) \\ = (H_3^2 - G_1G_2)Y^2 + 2(H_2H_3 - H_1G_1)YZ + (H_2^2 - G_1G_3)Z^2 + G_1D \\ = D(-g_3Y^2 + 2h_1YZ - g_2Z^2 + G_1) \\ = DA_1.\end{aligned}$$

(In this reduction, we have used again the general property of determinants referred to earlier.)

Since we originally arranged that $A_1 \geqslant 0$, it follows that our process does in fact give real X, and thus a totally real solution.

We are now able to consider the complete solution. We suppose suitable X, Y, Z have been chosen as above described, and the corresponding a_r and b_r evaluated. Then (33) gives (with the help of trigonometrical tables) the differences between the angles θ_1, θ_2, θ_3. We may take any one, say θ_1, arbitrarily. Then θ_2 and θ_3 are determined by (33), and so X' etc. by (32).

Of course, an alternative wholly algebraic solution is available, and perhaps it is worth while indicating the full algebraic process. We note first that from (33), we obtain

$$\begin{aligned}\sin(\theta_2 - \theta_3) &= \sqrt{(A_1/a_2a_3)}, \\ \sin(\theta_3 - \theta_1) &= \sqrt{(A_2/a_3a_1)}, \\ \sin(\theta_1 - \theta_2) &= \sqrt{(A_3/a_1a_2)}.\end{aligned}$$

Then

$$\begin{aligned}Y' = \sqrt{(ea_2)} \,.\, \cos\theta_2 &= \sqrt{(ea_2)} \,.\, \cos[\theta_1 - (\theta_1 - \theta_2)] \\ &= \sqrt{a_2}\left\{\frac{X'}{\sqrt{a_1}} \cdot \frac{b_3}{\sqrt{(a_1a_3)}} + \frac{X''}{\sqrt{a_1}} \cdot \sqrt{\left(\frac{A_3}{a_1a_2}\right)}\right\} \\ &= (b_3X' + \sqrt{A_3} \,.\, X'')/a_1\end{aligned}$$

and similarly for Z'.

We note also that the above-mentioned condition $A_1 \geqslant 0$ implies that in the Y, Z-plane the point (Y, Z) lies inside or on the ellipse

$$g_3Y^2 - 2h_1YZ + g_2Z^2 = G_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

Our solution is then as follows. Given a mass system, evaluate its second moments Mg_r and product moments Mh_r with respect to rectangular axes through its centroid, as in (24). Write down the determinant D as in (34) and evaluate the cofactors G_r and H_r of its elements g_r and h_r, and D itself.

Choose any point (Y, Z) within or on the ellipse (35), and evaluate

$$a_2 = g_2 - Y^2, \qquad a_3 = g_3 - Z^2, \qquad b_1 = h_1 - YZ, \qquad A_1 = a_2a_3 - b_1^2,$$

and so

$$X = [\sqrt{(DA_1)} - (H_3Y + H_2Z)]/G_1,$$

thus satisfying $\Delta = 0$. Evaluate the remaining elements of Δ:

$$a_1 = g_1 - X^2, \qquad b_2 = h_2 - ZX, \qquad b_3 = h_3 - XY,$$

and the minors

$$A_2 = a_3 a_1 - b_2^2, \qquad A_3 = a_1 a_2 - b_3^2.$$

Choose any masses m_s satisfying (25), and calculate c, d, e from (29). Now take arbitrary X' such that $X'^2 < ea_1$, and find $X'' = \sqrt{(ea_1 - X'^2)}$, and then in succession

$$Y' = (X'b_3 + X''\sqrt{A_3})/a_1, \qquad Y'' = (eb_3 - X'Y')/X'',$$
$$Z' = (X'b_2 - X''\sqrt{A_2})/a_1, \qquad Z'' = (eb_2 - Z'X')/X''.$$

We then have finally

$$x_1 = X/c, \qquad x_2 = dX' - cX, \qquad x_3 = \frac{m_4 X'' - m_2 dX'}{m_3 + m_4} - cX, \qquad x_4 = x_3 - X'',$$

and similarly for y_s and z_s.

It can be shown that all square-root signs involved in this solution, including those of c and d, may be taken arbitrarily + or −, except that $\sqrt{A_2}$ and $\sqrt{A_3}$ must be such as to make $\sqrt{(A_2 A_3)} = B_1 = b_2 b_3 - a_1 b_1$. Of course, the signs must be used consistently.

Then particles m_s placed at (x_s, y_s, z_s), $(s = 1, \ldots, 4)$, are equivalent to the given system.

The special case of equal particles, $m_s = M/4$, is characterised by the constants

$$c = \pm 1/\sqrt{3}, \qquad d = \pm 1/\sqrt{3}, \qquad e = 8.$$

We may summarise the results of the present section in the following theorem, which requires no further proof.

Theorem 6. *To any line, plane or space system of positive mass there are infinitely many equivalent real systems of* 2, 3 *or* 4 *positive particles respectively, and none with fewer particles. With these numbers, the total available mass may be distributed among the particles in any preassigned way, and the numbers of degrees of freedom of position then remaining are* 0, 1, 3 *respectively.*

In each case it is fairly obvious that equivalent systems exist with any number of particles exceeding the minimum number. That this is indeed so can be shown precisely in the following way. First construct any minimum-number equivalent particle system. This will contain at least two particles. Select a pair, say m_1 at the point P_1 and m_2 at P_2, and split up one of them, say m_1, into any two portions m_1' and m_1'', so that $m_1' + m_1'' = m_1$. While keeping m_1' at P_1, replace m_1'' and m_2 by a different equivalent two-particle system on the line P_1P_2. We have thus increased the number of particles by unity, while still having a system equivalent to the original system. This process may now be repeated as often as desired.

6. We are now in a position to prove the statement at the end of section 2, and to complete the proofs of Theorems 3 and 4.

Firstly, we show that two systems need not be equivalent even though they may have equal second moments about any *finite* number of lines, however large. (Theorem 3, on the other hand, shows that 6 lines suffice provided the systems are also known to have the same mass and centroid.) A trivial case occurs if the lines, even if infinite in number, all pass through a common point O. For if one system, with total mass M, has masses m (or dm) at points P, a new system with masses km (or $k\,dm$) at points Q which are in the lines OP but such that $OQ = OP/\sqrt{k}$, will clearly have the same second moments about all lines through O, but a total mass kM, so that by Theorem 1 it cannot be equivalent to the first system.

More generally we can show that given any number N (≥ 4) of lines L_i, and any one system, it is possible to construct a non-equivalent system having the same second moments about the given N lines. To do this, first construct an equivalent system having N particles m_s, at points P_s. This can always be done as explained at the end of section 5. Let p_{is} be the distances of the points P_s from the lines L_i, and p_{i0} those of an arbitrary point P_0, differing from all the other P_s.

Put mass increments δm_s at the points P_s ($s = 1, \ldots, N$) and a mass δm_0 at P_0, these quantities being chosen so as to satisfy the equations

$$\delta I_i = \sum_{s=0}^{N} \delta m_s \cdot p_{is}^2 = 0 \quad (i = 1, \ldots, N),$$

$$\delta M = \sum_{s=0}^{N} \delta m_s \qquad = \mu,$$

μ being an arbitrary non-zero quantity.

These $(N+1)$ equations can always be solved for the $(N+1)$ quantities δm_s provided the determinant of the coefficients does not vanish. Now, assuming this determinant does not vanish identically in the coordinates of P_0 (which would only occur for very special dispositions of the lines L_i and points P_s, and could then be avoided by constructing another equivalent system), we may choose P_0 in the first place so that it does not vanish.

The solutions δm_s ($s = 0, \ldots, N$) are proportional to μ. Choose the sign of μ so that δm_0 is positive, and make μ so small that $m_s + \delta m_s$ ($s = 1, \ldots, N$) are all positive. Then a system with $m_s + \delta m_s$ at P_s, and δm_0 at P_0 has the same moments I_i about the lines L_i, but a different total mass $M + \mu$, so that by Theorem 1 it cannot be equivalent to the original system.

To prove the second part of Theorem 4, namely that two plane systems may be non-equivalent even though they have the same mass and centroid and equal second moments about 2 given lines, which we may suppose pass through the centroid, let a given system with mass M and centroid G have second moments $M\kappa^2$, $M\lambda^2$ about any two given lines through G, and $M\mu^2$ about any third line through G. Calculate ρ as in (20), and choose any other value ρ' such that $\rho'^2 \leq \kappa^2\lambda^2$, and find the corresponding μ'^2 from (20). Then, as the discussion on plane systems in section 5 shows, it is possible to construct a real system (M, G) having second moments $M\kappa^2$, $M\lambda^2$, $M\mu'^2$ about the three lines, *i.e.* the same as for the given system about two of the lines, but not about the third. This proves the statement above.

Finally, to prove the second part of Theorem 3, we must show that two space systems may be non-equivalent even though they have the same mass and centroid and equal second moments about 5 given lines which again without loss of generality we may assume to pass through the centroid.

To see that this is so, take through a point G any 6 lines L_i, not common generators of a quadric cone. Take rectangular axes at G, and let the lines have direction-cosines (l_i, m_i, n_i). Then the determinant

$$\begin{vmatrix} l_i^2 & m_i^2 & n_i^2 & m_i n_i & n_i l_i & l_i m_i \end{vmatrix}$$

does not vanish, as we have seen in the proof of Theorem 3, first part, *i.e.* the determinant

$$\begin{vmatrix} m_i^2 + n_i^2 & n_i^2 + l_i^2 & l_i^2 + m_i^2 & m_i n_i & n_i l_i & l_i m_i \end{vmatrix}$$

does not vanish.

Now, if there are masses m_s at points $P_s(x_s, y_s, z_s)$ at distances p_{is} from the L_i,

$$\sum_s m_s p_{is}^2 = (m_i^2 + n_i^2)\Sigma m_s x_s^2 + \ldots + \ldots - 2m_i n_i \Sigma m_s y_s z_s - \ldots - \ldots.$$

It therefore follows that if second moments $I_i = \sum_s m_s p^2_{is}$ about the 6 lines are arbitrarily prescribed, it is possible to solve these 6 equations for corresponding expressions $\Sigma m_s x_s^2$, $\Sigma m_s y_s z_s$, etc., and the solutions are moreover continuous (linear) functions of the I_i.

Now, take any space system of positive particles m_s, with mass M and centroid at G, and calculate the second moments $I_i = \Sigma m_s p^2_{is}$. If we then solve for $\Sigma m_s x_s^2$, $\Sigma m_s y_s z_s$, etc., these quantities refer to a positive definite quadratic form. (Compare (34).) If now we replace I_6 by $I_6 + \delta I_6$, but leave $I_1, \ldots, I_5$ unaltered, we can again solve, and, by continuity, again get a positive definite form if δI_6 is small enough. This ensures that the method of section 5 can be used to obtain a real 4-particle system with mass M, centroid G, and second moments $I_1, \ldots, I_5, I_6 + \delta I_6$. This completes the proof of Theorem 3.

A. T.

1695. Henry'd do the books in an hour, or less even, and then sit around the whole rest of the day using up the firm's scratch paper doodling. When Mr. Rossiter had caught him at it, Henry'd said he was working problems—had some kind of dumb theory he was working on, he said. But Mr. Rossiter got some of the scratch paper out of the waste basket, and it was just plain doodling. Greek letters and stuff like that. He'd call Henry on it, asking him to explain some of the scratches. Henry had said (nearly as Mr. Rossiter could remember) that it was a time equation. Henry drew a picture and he said this picture was a vector t, and that the line meeting it was another vector t_2, and you turned the vector t_2 on itself by multiplying by the square root of minus one, and from there on Mr. Rossiter was lost. Of course, he knew why, because Henry was kidding him—giving him a line of double-talk and pretending it was maths.—Barbara Hunt, *A Little Night Music*. [Per Mr. G. H. Gratton-Guinness.]

1696. At ten, he had to stay after school to write on the blackboard a hundred times, "I am sorry I looked disrespectfully at my teacher." At 13 he had to repeat the performance, writing "A straight line is the shortest distance between two points", because he had questioned in class Euclid's right to make such an assumption. That mild discipline had ended in another beating because he had capped this century of axiomatic repetition by scrawling, "He still hasn't proved it," all over the bottom of the blackboard, and the teacher had sent him to the principal, who had written a note to his father, who had beaten him for his stupidity.—Barbara Hunt, *A Little Night Music*. [Per Mr. G. H. Grattan-Guinness.]

1697. Nam ut in libris de motu demonstratum est ab Aristotele, commensus quidam est temporis ad eam proportionem, quae est inter virtutem moventem et pondus seu molem mobilem, sive ponderis ad medium. Sed hic vis movens ad lucem movendam infinitam habet proportionem; quia luci nulla materia, quare neque pondus. Ita medium luci nihil resistit, quia lux materiâ caret per quam fiat resistentia. Ergo lucis infinita celeritas est. —J. Kepler, *Paralipomena* (1604), Chap. I, Prop. V. [Per Mr. B. A. Swinden.]

1698. The result is a "heads-I-win-tails-you-lose" argument. As if one were to claim that there are no musical references in Milton: because all music is basically mathematical; therefore all overtly musical themes are really comments on Euclid; therefore all musical references in Milton are in fact Euclidean; therefore there are no musical references in Milton.—"The Artist as Seismologist," by Father Martin Jarrett-Kerr, in *The New Outlook*, Spring, 1951. [Per Rev. A. F. Mackenzie.]

PLANE KINEMATICS.

By G. C. Steward.

1. *Historical.* That the problems of plane kinematics should have been studied for a long period is natural; they present themselves quite easily, and in simple fashion. At the end of the seventeenth century, and the beginning of the eighteenth century, De La Hire investigated roulettes, and to him also is due the introduction of the inflexion circle, as the aggregate of the points of a plane, subject to coplanar motion, which are instantaneously at inflexions of their paths. At a much later date, Ball (1) drew attention to that one point of the moving plane—upon the inflexion circle—which, in quite general motion, has four point contact with its approximately straight line path; or, has four consecutive positions in line.

In the eighteenth century, the engineer Watt designed a mechanism to generate approximately straight line motion, from non-linear motion, and his three-bar mechanism has since given rise to prolonged and extensive investigations. It is of interest to recall that doubts were expressed concerning the possibility of generating accurate straight line motion, by mechanical means; but such accurate generation was given by Peaucellier, and afterwards by Lipkine, a pupil of Tschebycheff. And, later, Kempe (2) shewed that *any* plane algebraic curve can be generated mechanically.

Throughout the nineteenth century, the properties of the three-bar mechanism attracted the attention of many writers, not least in England; we may mention the names of Tschebycheff, Schoenflies, Sylvester, Darboux (3), Clifford, Burmester, Cayley, Roberts, Kempe and many others. Roberts (4) gave his triple generation method of obtaining the three-bar curve, calling forth notice and treatment by Cayley (5); and perhaps we may see the culmination of such investigations, up to the present time, in a well-known paper by the late Dr. G. T. Bennett (6), on *The Three-Bar Sextic Curve*, in which the triple generation method of Roberts is taken as fundamental. The paper is comprehensive, and perhaps not easy to read; it is very compact and symmetrical, and several new properties of the curve are given, the path of a point rigidly attached to a three-bar mechanism—a trinodal, tricircular sextic curve. It would be difficult, we may feel, to imagine a more compact, symmetrical and powerful presentation. Reference may be made to the interesting comments of H. F. Baker (7), in his obituary notice of Dr. Bennett.

Mention should also be made of a paper by Morley (8), to which the attention of the present writer was drawn by Dr. Bennett; here the problem is handled in quite different, and very interesting, fashion, involving the use of elliptic functions.

The three-bar mechanism is fundamental for mechanisms in general. The papers mentioned above are all purely mathematical.

Studies of the properties of the three-bar mechanism have also prompted much in the general theory of plane kinematics. Tschebycheff (9), better known for his researches in the Theory of Numbers, wrote various papers, extending over some thirty-five years (1854–1888), upon this topic; his interest seems to have returned continually to the subject. His pupil Lipkine gave the accurate generation of straight line motion, and obtained a reward from the Russian Government for this work; Peaucellier's earlier discovery of it had apparently been overlooked. But Peaucellier himself appears to have been anticipated by Sarrut. It is still commonly believed, I think, that the mechanical generation of accurate straight line motion was first given by the invention, in 1864, of the Peaucellier cell. But in 1853 Sarrut (10) described a very simple mechanism for this purpose; and a report of Sarrut's

mechanism was given by Poncelet (11). For some interesting remarks upon the whole matter reference may be made to Bennett (12).

In connection with the three-bar mechanism, Tschebycheff himself discovered a certain result, afterwards generalised by Müller, and capable of still further generalisation ; to the effect that, when contact of a certain order has been obtained, with a straight line, there are then three other points of the mechanism, the paths of two having complete contact with circles, as guided by the mechanism, and that of the third five point contact with a third circle ; and, further, that these three points are collinear. This was generalised subsequently, by Müller, for quite general plane motion.

We may notice a treatment of the problems of plane kinematics due to Blaschke (13), in quite different fashion from any of the preceding.

2. *The Cardinal Points.* In plane kinematics we are concerned fundamentally with collineations in a plane, the special kinematical feature of which is that lengths are conserved. And this special feature implies that, of the three self-corresponding points arising in the general plane collineation, two coincide with the absolute points, and the third with the instantaneous centre, for infinitesimal displacements, or with the centre of rotation, for finite displacements. In an infinitesimal displacement, knowledge of the position of the instantaneous centre gives the direction of the displacement of each point fixed in the moving plane ; but the remaining properties of the path of such a point are not so determined. For example, the centre of curvature of the path, and also the centres of curvature of the several evolutes of the path, are not determined by the position of the instantaneous centre alone. In short, the information given by the knowledge of the location of the instantaneous centre is partial, and very incomplete.

In the elementary theory of geometrical optics, all the non-aberration properties of the general symmetrical optical system are completely determined by the knowledge of the three pairs of Cardinal Points, introduced by Gauss, of which only two pairs are independent. And it is the case, in the general theory of plane kinematics, that there exist certain points playing a somewhat similar rôle. For, we may define (14) an enumerable set of points $\mathcal{A}_1$, $\mathcal{A}_2$, ... $\mathcal{A}_n$, ... which, by their configuration, determine completely all the properties of the path of any point, or series of points, fixed in the moving plane. The configuration of this set of points is characteristic of the motion of the plane, and conversely ; and gives a very simple and compact synthesis of the whole realm of plane kinematics. Accordingly, we may refer to these points as the *Cardinal Points* of the motion. It is our purpose to present these cardinal points descriptively, and to show how all the kinematical properties are completely determined by their configuration.

3. Fundamentally, in plane kinematics, we have a duality ; for we are concerned with the relative motions of two planes, subject to coplanar displacement.

We assume, then, a plane p, and a superimposed plane ϖ, subject to relative coplanar displacement ; and, further, we assume that this relative displacement depends upon one parameter only, as in the usual mechanism, or linkage. It is convenient to take, as parameter, the angle ϕ between two lines, one fixed in each plane. And we use rectangular axes of coordinates $O(x, y)$, $\Omega(\xi, \eta)$ fixed in the planes p and ϖ respectively. Then we may show that for any point P_ϖ, fixed in ϖ, the corresponding complex coordinate $z(\phi)$, $\{\equiv x(\phi) + iy(\phi)\}$, relative to p, has, for variation of the parameter ϕ, an increment Δz, given by

$$\Delta z = \sigma_1 \Delta\phi + \sigma_2 (\Delta\phi)^2 + \sigma_3 (\Delta\phi)^3 + \ldots + \sigma_n (\Delta\phi)^n + \ldots , \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where the σ-coefficients are complex, and are given by

$$n!\,\sigma_n = i^n \mathcal{A}_n P_\varpi ;$$

and the points $\mathcal{A}_1$, $\mathcal{A}_2$, ... $\mathcal{A}_n$, ... are independent of P_ϖ.

It follows that all the properties of the path of the general point P_ϖ, relative to the plane p, depend only upon the configuration of the points $\mathcal{A}_1$, $\mathcal{A}_2$, ... $\mathcal{A}_n$, ... , so that the configuration of these *cardinal points*, as we may name them, is characteristic of the relative motion considered ; and, conversely, is itself completely determined by this relative motion, or displacement. Further, the points are fixed in neither plane, so that each point has two loci—one in each of the planes p and ϖ.

Dually, by considering a general point P_p, fixed in p, and its path relative to ϖ, we may similarly define a second enumerable set of cardinal points $\mathcal{A}_1'$, $\mathcal{A}_2'$, ... $\mathcal{A}_n'$, ... , having similar properties with respect to the path of P_p in ϖ. And, clearly, either enumerable set completely determines the other set.

We have analysed then the general displacement of either plane p, or ϖ, relative to the other plane, into the configuration of one or other of two enumerable sets of points, $\mathcal{A}_n$ or $\mathcal{A}_n'$, $n = 1, 2, 3, \ldots$. And it is the case that $\mathcal{A}_1$ and $\mathcal{A}_1'$ coincide, with the point commonly known as the instantaneous centre ; but, in general, for $n > 1$, $\mathcal{A}_n$ and $\mathcal{A}_n'$ do not coincide.

4. *On certain Special Configurations of the Cardinal Points.* In passing, it is of interest to consider briefly the configurations of the cardinal points in some simple mechanisms.

Thus, if the displacement of ϖ relative to p be one of pure rotation, that is, if there be a point P fixed in both planes, then all the cardinal points, of each set, coincide with P. Or, again, if P_ϖ describe a straight line l fixed in p, then $\mathcal{A}_2$, $\mathcal{A}_4$, ... $\mathcal{A}_{2n}$, ... lie upon l, while $\mathcal{A}_1$, $\mathcal{A}_3$, ... $\mathcal{A}_{2n+1}$, ... lie upon the perpendicular to l through P_ϖ. A particular case of this arises if a circle, fixed in ϖ, roll, without sliding, along a line fixed in p ; then $\mathcal{A}_1$ is at the point of contact, while $\mathcal{A}_2$, $\mathcal{A}_3$ ... $\mathcal{A}_n$, ... coincide with the centre of the circle. The cardinal points of the second set, $\mathcal{A}_n'$, lie upon the radius to the point of contact, being equally spaced along this line, at distances equal to the radius of the circle.

Another particular case of interest arises when, in addition, a second point P_ϖ', of ϖ, describes a straight line l', in p ; we have then *elliptical* displacement of ϖ relative to p, while the dual, that of p relative to ϖ, is *cardioid* displacement. Here $\mathcal{A}_1'$, $\mathcal{A}_1$ coincide with the intersection of the respective perpendiculars to l and l', at P_ϖ and P_ϖ', while also $\mathcal{A}_1$, $\mathcal{A}_3$, ... $\mathcal{A}_{2n+1}$, ... coincide, and $\mathcal{A}_2$, $\mathcal{A}_4$, ... $\mathcal{A}_{2n}$, ... fall at the point of intersection of l and l'.

From the general point of view, these cases are, of course, very special, and they give rise to highly degenerate forms of many of the curves and configurations of the succeeding paragraphs.

If the points P_ϖ and P_ϖ' describe circles in p, then we have the very important special case of *three-bar* motion of ϖ relative to p.

A case of interest also arises if $\mathcal{A}'_{n-1}\mathcal{A}_n'$ be perpendicular to $\mathcal{A}'_n\mathcal{A}'_{n+1}$, for all values of $n > 1$; we may refer to these cardinal points as forming then an *orthogonal* set. And we may shew that, if a set be orthogonal for any value of the parameter ϕ, then the set remains orthogonal for all values of ϕ. In other words, orthogonality is a property which is conserved throughout the relative motion. The kinematical implication of orthogonality is, in the preceding case, that there is a line λ fixed in ϖ and a curve c fixed in p, such that λ envelopes c, the point of contact being fixed upon λ.

5. *The Circle of Inflexions and the Ball-Point.* It follows from (1) that if the ratio of the first two coefficients, in the power series in $\Delta\phi$, be real, the point P_ϖ is at an inflexion of its path relative to the plane p ; or, that P_ϖ has three consecutive positions in line relative to p, all lines so arising passing through $\mathcal{A}_1$: and, also, that P_ϖ lies then upon the circle on $\mathcal{A}_1\mathcal{A}_2$ as diameter. Hence the name *circle of inflexions*. If, further, the ratio of the third

coefficient to the first, and so also to the second, be real, the path of P_ϖ has *four* consecutive positions in line relative to p. This is the Ball-point B, first indicated by Sir R. Ball, and it appears here as the foot of the perpendicular from $\mathcal{A}_3$ upon $\mathcal{A}_1\mathcal{A}_2$. And, generally, for unrestricted $\mathcal{A}_n$, this is the highest order contact possible with any straight line fixed in p. But if, as a special case, $\mathcal{A}_2\mathcal{A}_4$ be perpendicular to $\mathcal{A}_1\mathcal{A}_2$, there is *fifth* order contact at B: and, indeed, each additional higher order contact with such a straight line implies that an additional cardinal point $\mathcal{A}_n$ lies either upon $\mathcal{A}_1\mathcal{A}_2$, or else upon $\mathcal{A}_2\mathcal{A}_4$; and conversely.

If *all* the points, of either set, are so placed, that is, if, for example, $\mathcal{A}_{2n+1}$ lie upon $\mathcal{A}_1\mathcal{A}_2$ and $\mathcal{A}_{2n}$ lie upon $\mathcal{A}_2\mathcal{A}_4$ for all values of n, and if further these lines be perpendicular, we refer to the resulting configuration as a *rectangular* set of cardinal points. It is the case that rectangularity of either set is incompatible with rectangularity of the other set; and, further, that if a set be rectangular for one value of the parameter ϕ, then the set is rectangular for all values of ϕ.

There is, similarly, a dual inflexion circle, associated with $\mathcal{A}_1'$, $\mathcal{A}_2'$ and a dual Ball-point B', given in like manner by $\mathcal{A}_3'$; and, further, the two inflexion circles are of equal radii, and touch externally at $\mathcal{A}_1(\mathcal{A}_1')$.

The Ball-point, as a singular point, plays an important rôle, as related to the various curves which emerge subsequently.

6. *The Radii of Curvature of the Evolutes of the Path.* We are interested in the radius of curvature ρ_n of the nth evolute of the relative path of P_ϖ; and we may shew that this is given vectorially by an expression of the form

$$\rho_n = i^n \omega(n) P\mathcal{A}_1,$$

where $\omega(n)$ is real, and depends upon the first $n+2$ cardinal points, $\mathcal{A}_1$, $\mathcal{A}_2$, ... $\mathcal{A}_{n+2}$; and, a matter of some importance subsequently, depends linearly upon the co-ordinates of $\mathcal{A}_{n+2}$. The case $n=0$ corresponds, of course, to the ordinary radius of curvature of the path of P_ϖ. We are led to consider, then, the aggregate of points P_ϖ, for which $\rho_n = 0$, and these lie upon a curve $w(n)=0$, passing through the absolute points. These curves, of degree $4n-1$, are, for varying values of n, themselves of some interest. The curve $w(n)=0$ has a multiple point of order $2n$ at $\mathcal{A}_1$, and multiple points of order $2n-1$ at each of the absolute points, and passes through the Ball-point. This latter point is a singularity of the functions of position ρ_n.

But, interest centres in the intersections of these several curves, and, more particularly, in the intersections of $w(n)=0$ with $w(1)=0$; more particularly still in the possibility of coincidences between the sets of intersections with $w(1)=0$, for the several values of n. For example, such coincidences arise, for all values of n, in the important special case of three-bar motion.

7. *The Burmester Cubic.* The curve $w(1)=0$ is a rational circular cubic, touching $\mathcal{A}_1\mathcal{A}_2$ at $\mathcal{A}_1$, and having a node there, with perpendicular tangents. Referred to rectangular axes, $\mathcal{A}_1$ being the origin and $\mathcal{A}_1\mathcal{A}_2$ the axis of y, its equation may be written

$$(\alpha x + \beta y)(x^2+y^2) + xy = 0,$$

where α and β depend upon $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ only; or, parametrically,

$$x/-\beta = y/\epsilon = \epsilon^2/(\epsilon-\alpha)(\epsilon^2+\beta^2).$$

This curve, a strophoid, has long been familiar, but it arose, in this connection, I believe, in the writings of Burmester. The Ball-point, of parameter α', lies upon it, where α' is derived from $\mathcal{A}_2'$, $\mathcal{A}_3'$ as α from $\mathcal{A}_2$, $\mathcal{A}_3$; and there is also a double focus, of parameter $-\alpha$, which lies upon the curve.

There are various geometrical derivations of the rational cubic curve in a plane, but we may notice the following, as related to the present investigation, and as showing the genesis of the curve from the cardinal points alone.

Let any line parallel to $\mathcal{A}_1\mathcal{A}_2$ intersect the inflexion circle in p and q, and the line $\mathcal{A}_1\mathcal{A}_3$ in r; let $\mathcal{A}_1 p$, $\mathcal{A}_1 q$ intersect the line through r, parallel to $\mathcal{A}_1\mathcal{A}_3'$, in P and Q. Then P and Q lie upon the Burmester cubic.

Or, again, let (D), (E) be related ranges upon the lines $\mathcal{A}_1\mathcal{A}_2$, $\mathcal{A}_2\mathcal{A}_3$ respectively, such that $\mathcal{A}_1$, $\mathcal{A}_2$, D_∞ correspond to $\mathcal{A}_2$, $\mathcal{A}_3$, ∞, where $2D_\infty \mathcal{A}_2 = \mathcal{A}_3\mathcal{A}_1$; then the foot of the perpendicular from D, upon the line $E\mathcal{A}_1$, lies upon the Burmester cubic.

There is also a dual cubic, obtained by replacing α' for α, and $\beta'(=\beta)$ for β. It is the case that the centre of curvature of the relative path of any point upon either curve lies upon the other curve. And there are many other geometrical properties of interest.

8. *The Burmester Points.* Upon the Burmester cubic, the aggregate of points for which $\rho_1 = 0$, are certain points for which also $\rho_2 = 0$, given by $w(2) = 0$. Such points indicate *five* point contact of the path, with the corresponding circles of curvature.

But we may deal with the matter, in more general fashion, by considering the intersections of the curves $w(n) = 0$, and $w(1) = 0$: these intersections cluster at the multiple point $\mathcal{A}_1$ and the absolute points, and the Ball-point. But there are, in addition, certain "free" intersections, the positions of which depend upon the configuration of the cardinal points.

The curve $w(n) = 0$ depends upon all the cardinal points up to, and including, $\mathcal{A}_{n+2}$; and, further, depends linearly upon the coordinates of $\mathcal{A}_{n+2}$. If then the cardinal points $\mathcal{A}_1$, $\mathcal{A}_2$, ... $\mathcal{A}_{n+1}$ be regarded as assigned, the free intersections of the two curves appear as a linear series of sets of points, of freedom two, upon the curve $w(1) = 0$. It follows that two of these intersections may be assigned arbitrarily, the cardinal point $\mathcal{A}_{n+2}$ being thereby determined; and a particularly convenient determination arises if two of the free intersections coincide with the absolute points.

We may determine then an enumerable set of points F_4, F_5, ... F_{n+2}, ... such that, if $\mathcal{A}_4$ coincide with F_4, two of the free intersections of $w(2) = 0$ and $w(1) = 0$ coincide with the absolute points; and then, $\mathcal{A}_4$ being at F_4, $\mathcal{A}_5$ will coincide with F_5 if, similarly, two of the free intersections of $w(3) = 0$ and $w(1) = 0$ coincide with the absolute points; and so on, step by step. Further, we may take the points F_4, F_5, ... F_{n+2}, ... as new origins of coordinates for the successive cardinal points, and then the several equations, giving the corresponding sets of free intersections, take a particularly simple form.

In the case of the curves $w(2) = 0$ and $w(1) = 0$, we may write ξ, η as coordinates of $\mathcal{A}_4$, relative to F_4, and then the equation determining the free intersections of these two curves, the so-called *Burmester points*, takes the form

$$\Theta(\epsilon\,;\ \xi,\ \eta) \equiv (\epsilon + \alpha)(\epsilon + \alpha')(\epsilon^2 + \beta^2) + \epsilon(\epsilon\eta - \beta\xi) = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

There are, therefore, four Burmester points; and we regard (2) as the canonical form of the *Burmester quartic*.

There are, similarly, four dual Burmester points, upon the dual Burmester cubic; and the canonical quartic, being symmetrical in α and α', gives these also, where now, however, ξ, η are the coordinates of $\mathcal{A}_4'$ relative to F_4'.

9. *Müller's Theorem.* If one of the Burmester points falls at the Ball-point, so that $\Theta(\alpha') = 0$, it is clear, from the canonical form of the quartic, that the remaining three Burmester points are collinear; since the condition for the collinearity of three points, upon the cubic, is that the product of their parameters should be equal to $\alpha\beta^2$. This is the result found by Tschebycheff.

for the three-bar mechanism, and afterwards generalised, by Müller, for all mechanisms.

More completely : if the fourth cardinal point $\mathcal{A}_4$ lie upon the line $\mathcal{A}_2B$, then one Burmester point coincides with the Ball-point B, the remaining three being collinear, and all lines so arising, corresponding to varying positions of $\mathcal{A}_4$ upon $\mathcal{A}_2B$, pass through a point Q, upon $\mathcal{A}_1\mathcal{A}_2$, where

$$2(\alpha+\alpha')\,\mathcal{A}_1Q+1=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

It is of interest to notice that this point Q plays an identical rôle in the dual figure, since (3) is symmetrical in α and α'.

But more generally still : if $\mathcal{A}_4$ lie upon *any* arbitrarily assigned line, one Burmester point is thereby fixed, and the remaining three define a circle ; and the aggregate of such circles, arising from varying positions of $\mathcal{A}_4$ upon the chosen line, forms a coaxal system, with real points of intersection, one of which is always upon the cubic curve. In the case of the preceding paragraph, the system breaks up into the line infinity together with a pencil of lines through the point Q, which is then the position of the second limiting point of the coaxal system.

10. *Certain Generating Curves for the Burmester Points.* Again, from the canonical form of the quartic equation, $\Theta(\epsilon)=0$, it is clear that through an arbitrarily assigned $\mathcal{A}_4$ pass four lines, corresponding to the four Burmester points on the cubic, determined by $\mathcal{A}_4$. These lines, then, for varying $\mathcal{A}_4$, envelope a curve of class four, which is, in fact, a rational quintic $\mathcal{Q}$, given in line coordinates referred to F_4 as origin, by

$$\mathcal{Q}\equiv(\alpha u-\beta v)(\alpha' u-\beta v)(u^2+v^2)-u^2vw=0,$$

of which F_4 is the only finite focus. The curve $\mathcal{Q}$, which is symmetrical in α and α', and so plays an identical rôle in the dual configuration, determines then the Burmester points, for arbitrary $\mathcal{A}_4$; because if t be any one of the four tangents from $\mathcal{A}_4$, the perpendicular to t from $\mathcal{A}_1$ gives the Burmester point corresponding to t, by its single remaining intersection with the cubic, of which $\mathcal{A}_1$ is the node.

If $\mathcal{A}_4$ be upon $\mathcal{Q}$, then two Burmester points coincide ; if $\mathcal{A}_4$ be at one of the four cusps of $\mathcal{Q}$ then three Burmester points coincide ; while if $\mathcal{A}_4$ be at one of the two double points of $\mathcal{Q}$ the corresponding Burmester points coincide in pairs. It is the case that $\mathcal{Q}$ touches the line infinity at two points, one being an inflexion of the curve ; and if $\mathcal{A}_4$ be at either of these points, then all four Burmester points corresponding coincide at $\mathcal{A}_1$.

And, of the four real foci of this curve $\mathcal{Q}$, of class four, one only is in the finite part of the plane, the remaining three being the points of contact with the line infinity. From the preceding tangential equation, the one real and finite focus is the origin of coordinates, the point F_4 of §8 ; or, for the dual quintic curve $\mathcal{Q}'$, the point F_4'.

There are other properties of interest attaching to the curve $\mathcal{Q}$, which need not detain us here. But, in passing, we may mention that the pedal of $\mathcal{Q}$, with F_4 as origin, of equation

$$x^2y+(\alpha x-\beta y)(\alpha' x-\beta y)=0,$$

gives a very simple generation of the Burmester points, as depending upon an arbitrary $\mathcal{A}_4$. For, the circle on $F_4\mathcal{A}_4$ as diameter, $\mathcal{A}_4$ being arbitrary, intersects the pedal cubic in two points coincident at F_4, and, in addition, has four " free " intersections (D), the lines $\mathcal{A}_4D$ being tangents to $\mathcal{Q}$, so determining the corresponding Burmester points.

There is also a dual curve $\mathcal{Q}'$, having the same equation as $\mathcal{Q}$, since this is symmetrical in α and α', but the origin is now the dual point F_4'.

We have been considering the general kinematical linkage, assuming a quite arbitrary configuration of the cardinal points. But certain special cases occur—themselves of some generality—corresponding to particular configurations of the cardinal points. Thus, if $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, and so also $\mathcal{A}_2'$ and $\mathcal{A}_3'$, be all collinear, as for example, in the elliptical and cardioid motion of §4, then the Burmester cubic breaks up into the line $\mathcal{A}_1\mathcal{A}_2$, together with a certain circle, with centre upon this line. And the quintic curve $\mathcal{Q}$, as also the other configurations, degenerates in interesting manner. For $\mathcal{Q}$ becomes a coincident point-pair upon the line infinity, in the direction perpendicular to $\mathcal{A}_1\mathcal{A}_2$, together with the parabola

$$x^2 = 4\alpha\alpha'(y + \alpha\alpha'),$$

the focus of which is at F_4, and the axis parallel to $\mathcal{A}_1\mathcal{A}_2$. And the rational cubic, the pedal of $\mathcal{Q}$, reduces to

$$x^2(y + \alpha\alpha') = 0,$$

that is, the line through F_4 parallel to $\mathcal{A}_1\mathcal{A}_2$, taken twice, together with the tangent at the vertex of the parabola. We have then the familiar pedal property of the parabola, the focus being the pole.

11. *The Burmester Points as Generated by a Line and a Conic.* Yet again: it is evident, from the canonical form of the quartic $\Theta(\epsilon) = 0$, that the sets of Burmester points are coresidual with the Ball-point B. For, any conic through a set of such points intersects the cubic again in two further points, which are collinear with the Ball-point.

We may consider, then, an arbitrary line λ through the Ball-point, intersecting the cubic in two further points, P' and P''; and then the aggregate of conics, through P' and P'' and the various sets of Burmester points. Such an aggregate is a conic net $N(\lambda)$, depending upon λ. And, in passing, we may notice that each pair of conics of this net has a second common chord, other than $P'P''$, and all such common chords, for all pairs of conics of the net, have in common the point Q of §9. Indeed, with respect to P' and P'' as absolute points, the conics of the net are "circles", and Q is their radical centre.

In general, for an arbitrary λ-line through B, there is a cubic curve $U(\lambda)$, of which $N(\lambda)$ is the polar net; and the poles of the several conics are related collinearly with the corresponding $\mathcal{A}_4$. Further, the cubic $U(\lambda)$ breaks up into the line $P'P''$, together with a certain conic which touches QP' and QP'', at P' and P'' respectively.

The cubic $U(\lambda)$ then, the line and the conic taken together, may be regarded as a generating curve for the sets of Burmester points, upon the Burmester cubic. For, to an arbitrary $\mathcal{A}_4$ corresponds collinearly one point P, the polar conic of which, with respect to the cubic $U(\lambda)$, gives the four Burmester points, corresponding to $\mathcal{A}_4$, by its residual intersections, other than P' and P'', with the Burmester cubic.

We may choose the λ line to pass through the dual Ball-point B', and then this line would play identical rôles in the dual configurations. But it is not possible for the λ-conic, the remaining part of the generating cubic $U(\lambda)$, to be chosen similarly. For this would imply two intersections of the Burmester cubic and its dual, elsewhere than at $\mathcal{A}_1$ and the absolute points; and there are no such intersections.

It is convenient to choose the line $\mathcal{A}_3B$ as the λ-line, and then this is one of the three self-corresponding lines in the collineation between $\mathcal{A}_4$ and the pole P of the corresponding conic of the conic-net.

We may write then $U = LS$, where L is the λ-line, and S the λ-conic, given by

$$S \equiv \{4\beta(\alpha' - \alpha)x + \alpha'(\gamma + 2\gamma')y + 4\alpha'\}^2 + 3\alpha'\gamma^2(4\alpha x^2 + \alpha' y^2) = 0,$$

where

$$\gamma = \alpha + 3\alpha', \text{ and } \gamma' = 3\alpha + \alpha'.$$

Further, if we write

$$\omega_1 \equiv \gamma(\xi - \beta\gamma'),$$
$$\omega_2 \equiv 8\beta^2(\alpha' + \alpha) - \alpha'(\alpha' - \alpha)^2 - 4(\beta\xi - \alpha'\eta),$$
$$\omega_3 \equiv 2\beta(\alpha' - \alpha)\omega_1 + \alpha\gamma'\omega_2 - \alpha\alpha'\gamma^3,$$

the collineation between $\mathcal{A}_4(\xi, \eta)$ and the pole $P(X, Y)$ is given by

$$\alpha X/\omega_1 = \alpha' Y/\omega_2 = 2\alpha\alpha'/\omega_3.$$

The equation of the line L, or $B\mathcal{A}_2$, is given by

$$\omega_3 + \alpha'\gamma^2 = 0.$$

12. *Certain Special Nets N (λ).* For certain λ-lines the preceding investigation breaks down; because, for such, the corresponding net N (λ) contains at least one double line, and there is not then any generating cubic $U(\lambda)$. These cases are of some interest, for they correspond to certain special configurations of the Burmester points; indeed, the four points are then collinear, and, since they lie upon a cubic, there must be coincidences amongst them. And then also the fourth cardinal point $\mathcal{A}_4$ is upon the rational quintic Q, and indeed at a singular point of this curve.

These special cases are four in number, as is geometrically evident. For the Burmester cubic is of class four, and so two tangents can be drawn to it from the Ball-point, upon the curve—other than the tangent at this point. If the λ-line coincide with either tangent the points P' and P'', of §11, coincide, so that we have a degenerate case; and, similarly, if this λ-line pass through the node $\mathcal{A}_1$ of the curve. There is also a fourth case, if the λ-line pass through Q. For then the points P', P'' and Q are collinear, and clearly the system degenerates. The detailed investigation of these several cases is itself of some geometrical interest. We may, in passing, mention only one such, as illustration.

If the λ-line coincide with BT, one of the tangents to the cubic from the Ball-point B, other than the tangent at B, then the double line l of the conic-net passes through T, and is indeed the harmonic conjugate of $T\mathcal{A}_1$ with respect to the line pair TB, TQ. The cardinal point $\mathcal{A}_4$ is then at one of the two nodes of the quintic, and the four Burmester points coincide, in pairs, at the two remaining intersections of l with the cubic. In this case, the cubic U reduces to three lines passing through T, of which one is the line l.

13. *Conclusion.* The purpose of this article is to show that all the kinematical properties associated with the motion of two planes, subject to relative coplanar displacement, may be deduced from, and are completely characterised by, the configuration of either of two enumerable sets of cardinal points; and to indicate descriptively the manner of this dependence. And, perhaps, sufficient has been given for this purpose.

But many other matters of interest arise, in addition to those mentioned. We have considered only the intersections of the curves $w(2) = 0$ and $w(1) = 0$; there are many interesting properties arising from the similar intersections of the curves $w(n) = 0$, for higher values of n, and the curve $w(1) = 0$, depending upon the cardinal points $\mathcal{A}_3, \ldots \ldots \ldots \mathcal{A}_{n+2}$; and, in particular, there are properties of interest associated with the possibility of coincidences between these intersections, for the several values of n. Moreover, in this connection,

we have not considered here at all the very important special case of three-bar motion, or displacement.

It has been indicated, in §4, that if a set of cardinal points be orthogonal for one value of the parameter ϕ, the set remains orthogonal for all values of ϕ, and similarly for the rectangular set of §5. These are particular cases of the general property that the configuration of either set of these points, for a single value of ϕ, determines the configurations for all values of ϕ; that is, a single configuration of either set is completely characteristic of the generalised relative displacement, or motion, of the two planes, and so gives a simple and compact synthesis of the whole motion, with all its properties.

A set of cardinal points, therefore, throughout its changing configurations corresponding to varying values of the parameter ϕ—but all determined by the configuration corresponding to any single arbitrary value of ϕ—expresses the underlying unity of the relative motion of the two planes ϖ and p. These changing configurations possess, then, properties which are conserved throughout the relative motion. But, in the cases of orthogonal and rectangular sets there is, in addition, a certain conservation of a more special kind—a conservation of the *form* of the set itself. And we may seek other sets, possessing similarly some conservation of form: in this way, we are led to consider, in addition to orthogonal and rectangular sets, also *linear* and *spiral* sets, and, in a very special case, a *circular* set; each of which possesses conservation of form throughout the relative displacement, or motion.

And further, we have been concerned with the configuration of the cardinal points, and with the derived points and curves, associated with a single value of the parameter ϕ. But, as above, these points, and also these curves, depend upon ϕ, and for variation of the parameter, they have loci, and envelopes, in each of the planes p and ϖ. For example, each cardinal point, of each set, has two loci, one in each plane. In the case of the first cardinal point $\mathcal{A}_1(\mathcal{A}_1')$, we have the centrodes, with their familiar rolling property. For the loci of the remaining cardinal points, the properties are different, but all related intimately with the kinematical properties of the mechanism, or linkage.

Also, we have the *Ball-lines*, the loci of the Ball-points; and each Ball line is part of the envelope of the corresponding inflexion circle, the remaining part of the envelope being the corresponding centrode. And similar properties exist for the envelope of the Burmester cubics, and the loci of the Burmester points, to which we may refer as the *Burmester-lines*. The like considerations apply to each of the configurations and curves associated with a single value of the parameter.

And all these arise from what we may name the "generalised relative path" of the two planes. If we consider the various motions, all possible in the same generalised relative path, we have another set of points—the acceleration centre, and the centres of hyper-acceleration. The circle of inflexions, for example, is often defined by means of the instantaneous centre and the acceleration centre. But this circle itself is a property of the generalised relative path alone, while the acceleration centre depends upon the particular motion in that path. So it seems better to define first the inflexion circle, and then to regard the acceleration centre as a point lying upon it. Similarly, there are circles of higher orders, having for example $\mathcal{A}_n\mathcal{A}_{n+1}$ as diameter; then the corresponding hyper-centre lies upon this circle.

G. C. S.

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In this paper the present writer has given a more complete account of the introduction and use of the cardinal points.

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1699. He found himself able to argue, with a mathematical exactitude, that sooner or later the peoples of the world would merge into one human type. But he now wondered whether he had not fallen into the error of treating this as a mathematical problem, whereas, first and foremost, it was a human problem. The ciphers and digits for its solution were not mathematical symbols ; they were the tears and groans of men.—Robert Standish, *Mr. On Loong*, p. 98. [Per Mr. Sydney Thomson.]

1700. .... and therefore top performances in the event [triple jumping], have not travelled quite so far [as in high hurdling] along their asymptote towards their ultimate but indeterminate limit. . . .—*The Athlete* (Spring, 1950), p. 31. [Per Mr. H. Ap. Simon.]

MATHEMATICAL NOTES.

2275. *Chains of circles.*

In connection with Mr. Child's ingenious treatment * of the problem of re-entrant chains of circles each touching its neighbours and two given circles, there is a neat formula that surely must be known, though I have not seen it in the text-books, and perhaps it should be better known.

If two fixed circles have radii r_1, r_2 and δ is the distance apart of their centres, then the necessary and sufficient condition for the existence of a chain of n touching circles re-entering m circuits (m less than n and prime to it) is that

$$(r_1 - \mu r_2)(\mu r_1 - r_2) = \mu\delta^2 \quad \text{if } |\, r_1 - r_2 \,| > \delta,$$

or

$$(r_1 + \mu r_2)(\mu r_1 + r_2) = \mu\delta^2 \quad \text{if } r_1 + r_2 < \delta,$$

where

$$\mu = \tan^2\{(n+2m)\pi/4n\}.$$

The proof of these formulae, by inversion from the case where the given circles are concentric, should be well within the powers of a good scholarship candidate.

The case dealt with by Mr. Child has, in his notation, $r_1 = 1$, $r_2 = a$, $\delta = c$, and $|\, 1 - a \,| > c$, and if we use the relation $(1 - \mu a)(\mu - a) = \mu c^2$, we easily find that

$$L = 2\left\{1 - \frac{8\mu}{(1+\mu)^2}\right\}$$

$$= -2\cos(2m\pi/n).$$

It is therefore clear that Mr. Child's fraction which has $1 + Lx + x^2$ in its denominator will give on expansion a cycle of n coefficients which then recur.

The case of two equal fixed circles each of radius r gives a specially neat result and is worth mentioning: it is that the condition for a re-entrant train of touching circles is

$$2r = \delta\cos(m\pi/n).$$

H. Todd.

2276. *On some interesting sets of circles.*

A simple and powerful method of approach to Mr. J. M. Child's circles [*Math. Gazette* **32**, 52–58 (1948)] is by inverting the whole configuration with respect to a suitable centre, so as to simplify it. With non-intersecting, non-tangent circles one can thus obtain two *concentric* fixed circles S_1, S_1' instead of Mr. Child's S, S', and the sequence of circles touching the fixed circles and each its predecessor and successor then becomes trivially simple. If the radii of S_1, S_1' are a_1, a_1' respectively, then each circle of the sequence subtends an angle

$$\theta = 2\sin^{-1}\frac{a_1 - a_1'}{a_1 + a_1'} < \pi$$

at the common centre of S_1, S_1'. If this is a rational multiple of 2π,

$$\theta = (p/q)2\pi, \quad [2p < q, \quad (p, q) = 1],$$

then the sequence will recur: the qth circle touches the first, the $(q+1)$th coincides with the first; and meanwhile the circles have made p circuits. Hence we can have the "re-entry" case after a single circuit ($p = 1$) or after several ($p > 1$). Also q can have *any* integer value > 2, *e.g.* 4. It is to be noted that the behaviour of the sequence depends only on the relative sizes of S_1,

* *Mathematical Gazette*, XXXII, pp. 52–8.

S_1' (or the relative sizes and positions of S, S'), not on the position of the first circle of the sequence.

If θ is not a rational multiple of 2π, then the circles never re-enter. Their centres will then lie dense on the locus of centres, but will not exhaust that locus (by a well-known theorem on cardinals).

The case of two circles S, S' outside each other leads to the same configuration after suitable inversion; the case is amusing, though it adds nothing to the re-entry question. One or two of the circles of the sequence may degenerate to common tangents of S, S' (viz. to those touched by S, S' on the same side).

One can add an arbitrary element of ambiguity by allowing circles in the sequence which do not enclose both or neither of S, S', but enclose one and not the other. This can be done at any step in the sequence. If S, S' lie outside each other, then one can in this way also obtain the other two tangents.

The "simpler general theorem" (*l.c.*, p. 54), when S, S' touch each other, can also be easily dealt with by inversion. Choosing the point of contact as centre of inversion one obtains a pair of parallel lines; after which things become easy. A special case similar to Mr. Child's first case (*l.c.*, p. 56) is treated by Pappus by elementary (though *not* easy) arguments; and the beauty of his treatment leaves no room for improvement. A taste of it, to whet the appetite, can be had from Ivor Thomas: *Greek Mathematics*, vol. II (Loeb Classical Library, London and Cambridge, Mass., 1941) pp. 578 sqq.

If S, S' intersect, one can, by inversion, not attain central symmetry, but only axial symmetry; which may be some help. One could instead have recourse to complex valued coordinates and thus reduce it to the first case—not an attractive artifice, though. The simplest approach appears to be by inversion with respect to one of the intersections of S, S' as centre, when they will be transformed into intersecting straight lines, and one can again read off most of the information required. B. H. NEUMANN.

2277. *Construction of a quadrilateral* (Note 2156).

This note gives a geometrical solution of the problem of constructing a quadrilateral given the sides and the length x of the line joining the midpoints of two opposite sides, say a and c.

1. Construct the triangle PQR with sides b and d and median x. (The reader is invited to supply his own figure).

Draw $PQ = b$. With P and Q as centres and radius d describe arcs of circles; with P as centre and radius x describe a circle. With centre P and radius equal to the diameter of the circle 'x', draw an arc cutting that with centre Q at S. Join PS and let it cut the circle 'x' at T. Join QT and produce to meet the circle 'd' at R.

2. Draw $AB = a$. Bisect it at E. With E as centre and radius TQ from (1), describe a circle. With A and B as centres and radius $\frac{1}{2}c$ cut this circle at X and Y. With E as centre and radius x draw an arc of a circle, and with X as centre and radius d cut this circle at F. Then the triangle FXY is the triangle PQR of (1). Complete the parallelograms $FYBC$ and $FXAD$. Then $ABCD$ is the required quadrilateral.

EX could of course be calculated, since

$$b^2 + d^2 = 2x^2 + 2XE^2.$$

J. C. O'REILLY.

2278. *The envelope equation of the circular points.*

It is customary when beginning the study of trilinear coordinates to obtain the envelope equation of the circular points by considering the intersections of the line at infinity with the circumcircle of the triangle of reference, the equation of which is found by elementary geometrical considerations.

The following alternative method may be of interest to some readers.

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ denote the circular points I, J. As they lie on the line at infinity.

$$a\alpha_1 + b\beta_1 + c\gamma_1 = a\alpha_2 + b\beta_2 + c\gamma_2 = 0,$$

so that

$$(b\beta_1 + c\gamma_1)(b\beta_2 + c\gamma_2) = a^2\alpha_1\alpha_2,$$

or

$$bc(\beta_1\gamma_2 + \beta_2\gamma_1) = a^2\alpha_1\alpha_2 - b^2\beta_1\beta_2 - c^2\gamma_1\gamma_2 \quad \text{.....(i)}$$

The equation of the bisectors of the angle A of the triangle of reference is

$$(\beta - \gamma)(\beta + \gamma) = 0, \text{ or } \beta^2 = \gamma^2,$$

and I, J are conjugate with respect to this degenerate conic.

Hence $\beta_1\beta_2 = \gamma_1\gamma_2$ and so, similarly,

$$\alpha_1\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2 = \lambda, \text{ say.}$$

Thus from (i),

$$\beta_1\gamma_2 + \beta_2\gamma_1 = -2\lambda \cos A.$$

The envelope equation of I, J is

$$(\alpha_1 l + \beta_1 m + \gamma_1 n)(\alpha_2 l + \beta_2 m + \gamma_2 n) = 0,$$

or

$$\Sigma\alpha_1\alpha_2 l^2 + \Sigma(\beta_1\gamma_2 + \beta_2\gamma_1)mn = 0,$$

and by using the results obtained above, this becomes

$$l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0.$$

R. WALKER.

2279. *Note on alternating series.*

The sequence u_n is a sequence of positive numbers with limit zero. If the sequence is monotonic, then the series

$$S_p = \Sigma(-1)^{n-1}u_n{}^p$$

is convergent for any $p > 0$. If the sequence is not monotonic, all the series S_p may be divergent.

Given that S_1 is convergent, does it follow that S_p is convergent for some $p \neq 1$? The answer is no, as the following simple example shows. Take

$$u_1 = u_2 = 1, \quad u_{2m-1} = a_m + b_m + c_m, \quad u_{2m} = a_m \quad (m = 2, 3, \ldots)$$

where

$$a_m = 1/\log m, \quad b_m = (-1)^m a_m, \quad c_m = 2^{-m} (m \text{ odd}), \ 0 \ (m \text{ even}).$$

For this sequence S_1 is convergent but S_p divergent for every p different from 1.

More generally, we may take

$$a_m = f(m), \quad b_m = (-1)^m a_m, \quad c_m = \phi(m) (m \text{ odd}), \quad 0 \ (m \text{ even})$$

where $f(x)$ is a positive decreasing function of x with limit zero such that

$$\int^{\infty} f^p dx$$

is divergent for every $p > 0$, and $\phi(x)$ is a positive decreasing function of x such that

$$\int^{\infty} \phi^p dx$$

is convergent for every $p > 0$.

For,

$$S_1 = \Sigma b_m + \Sigma c_m,$$

the sum of two convergent series. On the other hand, we see by Cauchy's integral test that the sum of the first $2n$ terms of S_p behaves like

$$(2^{p-1}-1)\int_a^{2n}\{f(x)\}^p dx$$

for a convenient $a>0$ when $p>0$ and $p\neq 1$, so that S_p is divergent for all positive p except $p=1$.

It may be thought that the convergence of S_1 and S_p (for a given $p>1$) should ensure the convergence of S_r for $1<r<p$. But this is not the case. Take

$$a_m=m^{-1/p}(\log m)^{-\alpha}(\alpha>1/p),\quad b_m=(-1)^m a_m,\quad c_m=2^{-m}\text{ or }0.$$

Here S_1 and S_p are convergent but S_r is divergent for $1<r<p$. In this case S_q is absolutely convergent for $q\geqslant p$.

It would be interesting to know whether the convergence of S_1 and S_p involves the convergence of S_q for $q>p$. A. OPPENHEIM.

2280. *Contact of a Quadric and a Ruled Surface.*

In George Salmon's classical text book *A Treatise on the Analytic Geometry of Three Dimensions* it is stated (5th edition, volume II, p. 86) that " Given any generator of a ruled surface, we can describe a hyperboloid of one sheet, which shall have this generator in common with the ruled surface, and which shall also have the same tangent plane with that surface at every point of their common generator ".

Similar statements appear in other well known text books, *e.g.* Frost and Smith, and R. J. T. Bell's *Elementary Treatise on Coordinate Geometry of Three Dimensions*. In each case, it is either stated or implied that the quadric constructed is unique. Two examples in Bell for instance ask the reader to find the equation of *the* quadric which touches a given ruled surface along a given line. It is interesting that in every case the fact seems to have been missed that, while admittedly the quadric constructed has special properties which distinguish it from the others, there is in fact an infinity of quadrics with the stated property.

Salmon's proof is based on the statement that " if two ruled surfaces have three consecutive generators in common, they will touch along the first of these generators ", and the other proofs are mostly similar. But only two consecutive generators in common are necessary. Frost (*Solid Geometry*, p. 205) for example, takes three consecutive generators AA', BB', CC', P a point on AA' and R the point in which the plane PBB' cuts CC'. The tangent plane is then the limiting position of the plane AA' and PR. He continues " since the position of the tangent plane at a point P on the generator depends only on the consecutive generators BB', CC' ... ". But in fact it depends only on BB', for the plane of P and BB' itself determines R, not vice-versa.

Bell adds " Since the surface and conicoid have three consecutive generators in common, the shortest distance and angle between the given generator and a consecutive generator are the same for both " and deduces that the central point and parameter of distribution are the same for the generator regarded as on the surface or on the quadric. But only two consecutive generators in common are needed for this. The central point and parameter are the same for the surface or any of the infinity of quadrics mentioned above.

The existence of this infinite number of quadrics is not overlooked elsewhere. For example (in a book chosen more or less at random) Carnoy (*Cours de Géométrie Analytique*) proves (p. 484) " Il y a donc une infinité de surfaces du second ordre qui peuvent avoir le même plan tangent avec la surface gauche tout le long d'une génératrice, c'est-à-dire, qui se racordent avec la

surface en tours les points de cette droite." He makes no mention, however, of the particular conicoid taken by Salmon, etc. as the unique one. I can find no mention in these books of the combined facts, firstly, that there exists an infinity of quadrics touching the surface along the generator, and secondly, that of these, one which might be called an osculating quadric or quadric of closest fit, possesses the special property that its generator of the other system through any point on the given generator is the second inflexional tangent to the surface at the point.

Dr. L. S. Goddard, who has read this note, has suggested that it might be of interest to include here a proof ab initio of the preceding statement and proposes the following.

" Consider a given ruled surface, R and a given generator g of R. A quadric Q contains g if it contains three points of g that is, there are three (linear) conditions on Q in order that Q contains g.

If Q contains g, each plane of the pencil of planes through g is a tangent both to R and Q. Hence the pencil sets up a (1, 1) correspondence between the points of g, as follows. Let P be a given point on g. At P take the tangent plane w, to R and suppose w is a tangent to Q at a point P' (on g). Then P and P' are corresponding points. Now there are two self-corresponding points, so that any quadric containing g touches R at two points on g.

Exceptionally, if there are three self-corresponding points, then all points of g are self-corresponding, and Q touches R at every point of g. Thus the conditions on Q for the latter are (i) Q contains g, (ii) Q touches a given plane at a given point of g (the given plane being the tangent plane to R at the point). Now (i) and (ii) each impose three linear conditions on Q so that, since the dimension of the total system of quadrics is nine, we have: *there is a linear ∞^3-system of quadrics touching a given ruled surface R at every point of a given generator of R.*

Consider now the system of inflexional tangents to R at the points of g. These are the lines which meet g and two consecutive generators g', g'' of R as g', $g'' \to g$; hence these lines define a unique quadric Q' which touches R at every point of g, and therefore belongs to the ∞^3-system mentioned above."

Salmon begins by constructing this quadric Q' and so misses the ∞^3-system.

N. D. HAYES.

2281. *The kinetic energy of a rigid body.*

The following proof of the standard formula does not appear in any book which I can find, and seems to introduce a number of valuable teaching points though it is, of course, of no value mathematically.

If the centroid of a rigid body has a velocity V, and if the angular velocity of the body is ω, then a point distance x from G along a line through G perpendicular to the direction of V has a velocity $V - x\omega$. The instantaneous centre of rotation is therefore situated along this line at a distance V/ω from G.

If I_O, I_G denote the moments of inertia about axes through the instantaneous centre and centroid respectively, then the kinetic energy of the body is

$$\begin{aligned} \tfrac{1}{2} I_O \omega^2 &= \tfrac{1}{2}(I_G + Mx^2)\omega^2 \\ &= \tfrac{1}{2} I_G \omega^2 + \tfrac{1}{2} M (x\omega)^2 \\ &= \tfrac{1}{2} I_G \omega^2 + \tfrac{1}{2} M V^2. \end{aligned}$$

D. A. QUADLING.

2282. *The lens formula.*

The following proof of the lens formula was suggested to me by an acquaintance some years ago. The assumptions made, which seem to be no stronger than those made in the usual geometrical proofs, are :

(1) To every object there corresponds a unique image, their distances being related by an algebraic (1, 1) correspondence.

(2) Light rays are reversible.

(3) As the distance of the object from the lens tends to zero, so does the distance of the image.

The value of v corresponding to an infinite value of u is denoted by f. Then by (1) the relation between u and v is of the form

$$auv + bu + cv + d = 0$$

By (2) $b = c$, and by (3) $d = 0$ and hence

$$\frac{1}{u} + \frac{1}{v} = -\frac{a}{b}.$$

But when $u = \infty$, $v = f$.

Thus

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

D. A. QUADLING.

2283. *On Note* 2181.

In reply to Mr. Butterworth's enquiry on p. 299 of *Gazette* 310, the matter is covered in volume II of the *Advanced Algebra* which he quotes.

It occurs in an exercise (XIVc, 27) but this question is set in parts, so that a scholarship student may be expected to answer it for himself. In case of need, there is supplementary information in the *Hints* volume. The argument may be summarised as follows.

If (a_n) is monotone and diverges to $+\infty$, and $S_r = a_1 - a_2 + \ldots$ to r terms,

(1) $S_{2n+1} = a_1 + (a_3 - a_2) + \ldots + (a_{2n+1} - a_{2n})$ which increases and tends to L_1 or to $+\infty$.

(2) Similarly S_{2n} tends to L_2 or to $-\infty$.

(3) If L_1 and L_2 both exist, $a_{2n+1} = S_{2n+1} - S_{2n} < (L_1 + \epsilon_1 - L_2 + \epsilon_2)$ when n is sufficiently large, and so a_{2n+1} does not tend to $+\infty$. It can also be argued that it is often more instructive for a scholarship student to use an *ad hoc* method than merely to quote a result like D'Alembert's ratio-test with which he has been supplied.

C. V. DURELL and A. ROBSON.

2284. *Some criteria for the infinite oscillation of alternating series.*

In Note 2181 I. B. Butterworth raises the interesting problem of finding elementary tests for the infinite oscillation of alternating series and proves

TEST 1. *If* $a_r > 0$ *and* $a_{r+1}/a_r \to k > 1$ *then* $\Sigma(-1)^r a_r$ *oscillates infinitely.*

This test suffices however to determine only very rapid oscillations and it fails when it is applied, for instance, to the series

$$1^p - 2^p + 3^p - 4^p + 5^p - \ldots$$

no matter how large a positive integer p may be.

The simplest test for infinite oscillation of the rate $1 - 2 + 3 - 4 + 5 - \ldots$ appears to be

TEST 2. $\Sigma(-1)^n a_n$ *oscillates infinitely if* $a_{n+1} - a_n$ *is ultimately non-negative and does not tend to zero.*

We remark first that Test 2 contains Test 1 for if $a_{n+1}/a_n \to k > 1$ and $a_n > 0$ then there is a p such that, for $n \geqslant p$, $a_{n+1} - a_n > \frac{1}{2}(k-1)a_n > 0$, and therefore for $n \geqslant p$, $a_{n+1} - a_n > \frac{1}{2}(k-1)a_p$, which proves that the difference $a_{n+1} - a_n$ is positive and does not tend to zero.

To prove Test 2 it suffices to observe that if $a_{n+1} - a_n$ is non-negative and does not tend to zero then at least one of the differences $a_{2n+1} - a_{2n}$, $a_{2n} - a_{2n-1}$ does not tend to zero and therefore at least one of the sums $a_0 + (a_2 - a_1) + \ldots +$

$(a_{2n} - a_{2n-1})$, $(a_1 - a_0) + (a_3 - a_2) + \ldots + (a_{2n+1} - a_{2n})$ diverges to infinity ; hence if $A_n = a_0 - a_1 + a_2 - \ldots + (-1)^n a_n$ then A_{2n} is positive, A_{2n+1} is negative and either $A_{2n} \to \infty$ or $A_{2n+1} \to -\infty$ and so $\Sigma(-1)^n a_n$ oscillates infinitely.

As an application of Test 2 we take the series

$$1^p - 2^p + 3^p - 4^p + \ldots$$

where p is a positive integer ; if $a_n = n^p$, then

$$a_{n+1} - a_n = (n+1)^p - n^p \geqslant 1$$

so that $a_{n+1} - a_n$ does not tend to zero.

Test 2 fails to prove the infinite oscillation of the series

$$\log 1 - \log 2 + \log 3 - \log 4 + \log 5 - \ldots$$

A rather crude test for this rate of oscillation is

TEST 3. *$\Sigma(-1)^n a_n$ oscillates infinitely if $\underline{\lim}\, n(a_{n+1} - a_n) > 0$.*

For if $\underline{\lim}\, n(a_{n+1} - a_n) > 0$ then there is a positive α and an integer p such that, for $n \geqslant p$,

$$a_{n+1} - a_n > \alpha/n$$

which proves (by comparison with the series $\Sigma(1/n)$) that both A_{2n} and $-A_{2n+1}$ tend to infinity.

The application of Test 3 to the series with $a_n = \log n$ proves the infinite oscillation of the series, since $n(\log(n+1) - \log n) = \log(1 + 1/n)^n \to 1$. The test succeeds also with series of the type $\Sigma(-1)^{n+1} n^{1/2}$.

A slight refinement of Test 3 is

TEST 4. *If $a_{n+1} - a_n$ is positive and monotonic decreasing and $n(a_{n+1} - a_n)$ does not tend to zero, then $\Sigma(-1)^n a_n$ oscillates infinitely.*

For by Abel's theorem, since $a_{n+1} - a_n$ is monotonic decreasing the series $\Sigma(a_{2n+1} - a_{2n})$, $\Sigma(a_{2n} - a_{2n-1})$ cannot both converge unless $n(a_{n+1} - a_n)$ tends to zero.

Each of the foregoing tests derives from a test for divergence of one of the positive series $\Sigma(a_{2n+1} - a_{2n})$, $\Sigma(a_{2n} - a_{2n-1})$; by comparing these series in turn with the known divergent series whose general terms are respectively $1/n \log n$, $1/n \log n \log\log n$, and so on, we may obtain a succession of tests for infinite oscillation, each more delicate than the last, and each test may be refined (along the lines of Test 4) by imposing a condition of the form

$$d_n(a_{n+1} - a_n) \text{ steadily decreases}$$

where d_n takes in turn the values n, $n \log n$, $n \log n \log\log n$, and so on.

We conclude with a kind of Gauss test for infinite oscillation.

If

$$a_{n+1} - a_n = \alpha + \beta/n + \theta_n/n^{1+\lambda}$$

where θ_n is bounded and $\lambda > 0$, then the series $\Sigma(-1)^n a_n$ oscillates infinitely if and only if $\alpha > 0$, or $\alpha = 0$ and $\beta > 0$.

The case $\alpha > 0$ is established by Test 2, and the case $\alpha = 0$, $\beta > 0$ by Test 3. If $\alpha < 0$ or $\alpha = 0$, $\beta < 0$ then a_n is steadily decreasing and so $\Sigma(-1)^n a_n$ lies between a_0 and $a_0 - a_1$; if $\alpha = 0$ and $\beta = 0$ then $\Sigma|a_{n+1} - a_n|$ is convergent so that both A_{2n} and A_{2n+1} are convergent and the series $\Sigma(-1)^n a_n$ is bounded.

R. L. G.

2285. *A Note on Multiple Exponentials.*

Professor J. E. Littlewood, in the course of an article on " Large Numbers " in the *Mathematical Gazette* for July 1948, briefly discusses multiple exponentials, *i.e.* functions of the form : " a to the power of b to the power of c ... etc. etc." Such a function will here be represented by the symbol : $\{a, b, c \ldots\}$

It is not perhaps immediately obvious that an infinitely repeated exponential $\{d, d, d \dots\}$ tends to a finite limit for a range of values of d greater than unity.

It will be shown below that if $1 \leqslant d \leqslant e^{1/e}$ then $\{d_1, d_2 \dots d_n\}$ tends to a limit as the subscript n, representing the number of exponents, tends to infinity, Moreover, this limit will not exceed e where e, as usual, represents the base of natural logarithms.

Consider the equation

$$\mu = d^{\mu}, \quad (= \{d, \mu\}) \qquad \dots\dots(1)$$

restricted to values of $\mu \geqslant 1$.

Raising d to equal powers, we see that this equation implies

$$\mu = \{d, \mu\} = \{d, d, \mu\} = \dots = \{d_1, d_2, \dots d_n, \mu\} \qquad \dots\dots(2)$$

however large n may be.

Now since we can re-write (1) as

$$d = \mu^{1/\mu} \qquad \dots\dots(3)$$

and since $\mu \geqslant 1$ we have the inequality

$$1 \leqslant d \leqslant \mu \qquad \dots\dots(4)$$

and, therefore,

$$\{d_1 \dots d_n\} = \{d_1 \dots d_{n-1}, d_n\} \leqslant \{d_1 \dots d_{n-1}, \mu\} = \mu \qquad \dots\dots(5)$$

It remains to determine the range of d possible under (1). Re-writing this equation again in the form (3) and differentiating with regard to μ, we see that

$$D_{\mu} d = \frac{\mu^{1/\mu}}{\mu^2} (1 - \log \mu).$$

Equating the right hand side with zero, we find that d maximises when $\mu = e$, that is to say when $d = e^{1/e} = 1{\cdot}445$.

Therefore, provided $1 \leqslant d \leqslant e^{1/e}$, (5) holds and $\{d_1 \dots d_n\} \leqslant \mu \leqslant e$ however large n may be.

It is not difficult to extend this proof to show the convergence of $\{d_1 \dots d_n\}$ for the range $0 \leqslant d \leqslant e^{1/e}$.

E. DE ST. Q. ISAACSON.

2286. *On the condition that the general equation of the second degree should represent a pair of straight lines.*

For the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \dots\dots(1)$$

to be a pair of straight lines, the condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \qquad \dots\dots(2)$$

is necessary and sufficient.

1. *Suppose that* $h^2 - ab \neq 0$.

Let $$ax^2 + 2hxy + by^2 \equiv (\alpha x + \beta y)(\gamma x + \delta y),$$

and $$2gx + 2fy \equiv \kappa(\alpha x + \beta y) + \lambda(\gamma x + \delta y).$$

Then $$a = \alpha\gamma,\ b = \beta\delta,\ 2g = \kappa\alpha + \lambda\gamma,\ 2f = \kappa\beta + \lambda\delta,$$

and $$2h = \alpha\delta + \beta\gamma,$$

where the latter step can be made since $h^2 - ab \neq 0$ and so $\alpha\delta - \beta\gamma \neq 0$. Thus (1) becomes

$$(\alpha x + \beta y)(\gamma x + \delta y) + \kappa(\alpha x + \beta y) + \lambda(\gamma x + \delta y) + c = 0 \qquad \dots\dots(3)$$

The following argument is reversible :

Assuming (2) to be true, we substitute in it the values of a, b, f, g, h in terms of $\alpha, \beta, \gamma, \delta, \kappa, \lambda$ and obtain

$$4\alpha\beta\gamma\delta c + (\beta\kappa + \delta\lambda)(\alpha\kappa + \gamma\lambda)(\alpha\delta + \beta\gamma) - \alpha\gamma(\beta\kappa + \delta\lambda)^2 - \beta\delta(\alpha\kappa + \gamma\lambda)^2 - c(\alpha\delta + \beta\gamma)^2 = 0$$

which reduces to

$$(\kappa\lambda - c)(\beta^2\gamma^2 - 2\alpha\beta\gamma\delta + \alpha^2\delta^2) = 0$$

that is,

$$(\kappa\lambda - c)(\beta\gamma - \alpha\delta)^2 = 0$$

or

$$4(\kappa\lambda - c)(h^2 - ab) = 0.$$

Since $h^2 - ab \neq 0$, we have $\kappa\lambda = c$, that is

$$1 : \kappa = \lambda : c,$$

and hence (3) factorises " in pairs ".

2. *Suppose that* $h^2 - ab = 0$.

Let $p = \sqrt{a}$, $q = \sqrt{b}$, with signs chosen so that $pq = h$. Then (1) becomes

$$(px + qy)^2 + 2gx + 2fy + c = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and (2) becomes

$$h^2c + 2fgpq - p^2f^2 - q^2g^2 - ch^2 = 0,$$

that is

$$(pf - qg)^2 = 0,$$

or

$$pf = qg. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The following argument is reversible :

Assuming (5) to be true, $p : q = g : f$, and so $2gz + 2fy$ can be written as $2r(px + qy)$. Thus (4) is factorisable as a quadratic in $(px + qy)$.

B. D. PRICE.

2287. *Completely factorisable quadratics*

1. The quadratic (a, b, c integers) .

$$ax^2 + bx + c$$

may be said to be " completely factorisable " if all the four quadratics

$$ax^2 \pm bx \pm c$$

are factorisable. For example, $3x^2 \pm 10x \pm 8$ is such a set and provides good practice for capable pupils.

2. Factorising $3x^2 \pm 10x \pm 8$ presents the same problems as factorising $x^2 \pm 10x \pm 24$. The quadratics $ax^2 \pm bx \pm c$ are factorisable if and only if $x^2 \pm bx \pm ac$ are factorisable, and without loss of generality we may put $a = 1$.

3. Any solution may be simply extended thus :

$$x^2 \pm 5x \pm 6$$

$$x^2 \pm 5xy \pm 6y^2$$

Put $y = 2$: $\qquad x^2 \pm 10x \pm 24.$

Conversely it can be easily proved that if b and c have a prime common factor (p.c.f.) then c contains it twice and the factors can be removed, leaving another solution. If b and c have no p.c.f. we may term the solution " prime ". Prime solutions can then be extended to non-prime solutions thus :

$$x^2 \pm 5x \pm 6$$

$$x^2 \pm 10x \pm 24$$

$$3x^2 \pm 10x \pm 8.$$

4. Let $x^2 \pm bx \pm c$ be a solution. Then if α, β are the roots of $x^2 - bx + c = 0$ (all letters are integers) and if γ, δ are the positive roots of $x^2 \pm bx - c = 0$, $\gamma > \delta$, the required conditions are

$$\alpha + \beta = b,\ \alpha\beta = c,\ \gamma - \delta = b,\ \gamma\delta = c. \quad \ldots\ldots(1)$$

Since $\alpha\beta = \gamma\delta$,

$$\alpha/\gamma = \delta/\beta = \eta/\theta, \text{ say,}$$

where η, θ have no p.c.f. Hence

$$\alpha = \kappa\eta,\quad \gamma = \kappa\theta,\quad \delta = \lambda\eta,\quad \beta = \lambda\theta. \quad \ldots\ldots(2)$$

Since $\alpha + \beta = \gamma - \delta$,

$$\kappa\eta + \lambda\theta = \kappa\theta - \lambda\eta,$$

whence

$$\eta/\theta = (\kappa - \lambda)/(\kappa + \lambda). \quad \ldots\ldots(3)$$

Since η, θ have no p.c.f.,

$$\mu\eta = \kappa - \lambda,\quad \mu\theta = \kappa + \lambda. \quad \ldots\ldots(4)$$

Thus

$$2\kappa = \mu(\theta + \eta),\quad 2\lambda = \mu(\theta - \eta). \quad \ldots\ldots(5)$$

5. Suppose that the solution is prime. Then b and c have no p.c.f. Referring to (1), α and β can have no p.c.f., otherwise it would appear in b and c. Referring to (2), κ and λ likewise have no p.c.f. Finally, referring to (5), as κ and λ are co-prime μ cannot have factors in κ and λ, hence $\mu = 1$ or $\mu = 2$.

If $\mu = 1$, then by (2) and (4)

$$\alpha = \kappa\eta = \kappa(\kappa - \lambda),\quad \beta = \lambda\theta = \lambda(\kappa + \lambda),$$
$$b = \alpha + \beta = \kappa\eta + \lambda\theta = \kappa^2 + \lambda^2,$$
$$c = \alpha\beta = \kappa\eta\lambda\theta = \kappa\lambda(\kappa^2 - \lambda^2).$$

If $\mu = 2$, (5) becomes

$$\kappa = \theta + \eta,\quad \lambda = \theta - \eta,$$
$$b = \alpha + \beta = \kappa\eta + \lambda\theta = \theta^2 + \eta^2,$$
$$c = \alpha\beta = \kappa\eta\lambda\theta = \theta\eta(\theta^2 - \eta^2).$$

Thus whether $\mu = 1$ or $\mu = 2$ the same functions emerge for b and c.

6. We have proved that if $x^2 \pm bx \pm c$ is a prime solution, then integers κ and λ exist such that

$$b = \kappa^2 + \lambda^2,\quad c = \kappa\lambda(\kappa^2 - \lambda^2). \quad \ldots\ldots(6)$$

We now find the conditions to impose on κ and λ to ensure prime solutions.

7. Whatever κ and λ (6) gives a solution, not necessarily prime. Taking all upper or all lower signs

$$x^2 \pm (\kappa^2 + \lambda^2)x + \kappa\lambda(\kappa^2 - \lambda^2) = \{x \pm \lambda(\kappa + \lambda)\}\{x \pm \kappa(\kappa - \lambda)\},$$
$$x^2 \pm (\kappa^2 + \lambda^2)x - \kappa\lambda(\kappa^2 - \lambda^2) = \{x \pm \kappa(\kappa + \lambda)\}\{x \mp \lambda(\kappa - \lambda)\}.$$

8. We proceed to find the values of κ, λ which generate non-prime solutions. Let b, c have a p.c.f. p. Referring to (6), then there are three possibilities:

(*a*) κ and $\kappa^2 + \lambda^2$ contain p, whence κ and λ contain p.
(*b*) λ and $\kappa^2 + \lambda^2$ contain p, whence κ and λ contain p.
(*c*) $\kappa^2 - \lambda^2$ and $\kappa^2 + \lambda^2$ contain p.

In the last case, let

$$\kappa^2 + \lambda^2 = ps,\quad \kappa^2 - \lambda^2 = pt.$$

Then $2\kappa^2 = p(s + t)$, $2\lambda^2 = p(s - t)$.

Whence either $p = 2$, or κ and λ contain p. Examining (6), if $p = 2$, b and c are both even, and obviously κ and λ are either both even ar both odd, and conversely.

9. The complete solution for prime cases is therefore

$$x^2 \pm (\kappa^2 + \lambda^2)x \pm \kappa\lambda(\kappa^2 - \lambda^2),$$

(i) $\kappa > \lambda$, (ii) $\kappa - \lambda$ is odd, (iii) κ and λ have no common factor.

10. The prime cases with coefficients less than 100 are

$$\kappa = 2,\ \lambda = 1 \ :\ x^2 \pm 5x \pm 6\ ;$$
$$\kappa = 3,\ \lambda = 2 \ :\ x^2 \pm 13x \pm 30\ ;$$
$$\kappa = 4,\ \lambda = 1 \ :\ x^2 \pm 17x \pm 60\ ;$$
$$\kappa = 4,\ \lambda = 3 \ :\ x^2 \pm 25x \pm 84.$$

B. D. PRICE.

2288. *Determination of the real foci of a central conic.*

1. In two of the standard methods of finding the foci of a central conic, a quadratic equation is obtained at an early stage, one root of which leads to the real foci and the other to the imaginary foci. It is the purpose of this Note to show how to choose the root which leads to the real foci without using trial and error methods.

2. *Method* 1. If the point equation of the conic is

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

then its line equation is

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

and any conic confocal with it is

$$\Sigma + \lambda(l^2 + m^2) = 0$$

If this degenerates, it represents a pair of foci. The condition for this is

$$\begin{vmatrix} A+\lambda & G & G \\ H & B+\lambda & F \\ G & F & C \end{vmatrix} = 0,$$

that is,

$$C\lambda^2 + \Delta(a+b)\lambda + \Delta^2 = 0, \qquad (1)$$

which has two real roots.

Now, in general, if a quadratic form in three homogeneous variables, say,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

factorises, the conditions for real factors are

$$f^2 - bc > 0, \quad g^2 - ca > 0, \quad h^2 - ab > 0,$$

omitting the cases where any of these expressions is zero. Of these conditions, any one implies the other two, since, for example

$$\begin{aligned}(g^2 - ca)(h^2 - ab) &= g^2h^2 + a(abc - bg^2 - ch^2) \\ &= g^2h^2 + a(af^2 - 2fgh), \text{ since } \Delta = 0, \\ &= (af - gh)^2 > 0.\end{aligned}$$

In our case, the three conditions are

$$F^2 - C(B+\lambda) > 0, \qquad (2)$$
$$G^2 - C(A+\lambda) > 0, \qquad (3)$$
$$H^2 - (A+\lambda)(B+\lambda) > 0, \qquad (4)$$

one of which is sufficient.

3. *Method* 2. We try to express the point equation in the form

$$S' \equiv (x-\alpha)^2 + (y-\beta)^2 - e^2(lx+my+n)^2/(l^2+m^2)$$
$$\equiv (x-\alpha)^2 + (y-\beta)^2 - (px+qy+r)^2 = 0,$$

so that (α, β) is a focus and $px+qy+r=0$ is the corresponding directrix. We must have, therefore,

$$\mu S + S' \equiv 0,$$

for some value of μ. This gives the six equations

$$\mu a + 1 = p^2, \quad \ldots\ldots(5); \qquad \mu g - \alpha = pr, \quad \ldots\ldots(8)$$
$$\mu h = pq, \quad \ldots\ldots(6); \qquad \mu f - \beta = qr, \quad \ldots\ldots(9)$$
$$\mu b + 1 = q^2, \quad \ldots\ldots(7); \qquad \mu c + \alpha^2 + \beta^2 = r^2 \quad \ldots\ldots(10)$$

From (5), (6), (7) we get

$$(\mu a + 1)(\mu b + 1) = \mu^2 h^2,$$

that is,

$$C\mu^2 + (a+b)\mu + 1 = 0. \quad \ldots\ldots(11)$$

If the roots of (11) are μ_1, μ_2, then

$$(\mu_1 a + 1)(\mu_2 a + 1) = \mu_1\mu_2 a^2 + (\mu_1 + \mu_2)a + 1$$
$$= \{a^2 - a(a+b) + C\}/C$$
$$= -h^2/C.$$

For a hyperbola, the two values of $\mu a + 1$ are therefore of the same sign, which is easily shown to be positive, so that both values of p^2 are positive. For an ellipse, the two values of $\mu a + 1$ are of opposite signs, so that we must choose that one with the positive sign.

Having found μ from (11), we obtain p from (5), choosing the positive square root (if we choose the negative we merely get $-p$, $-q$, $-r$ instead of p, q, r), then q from (6), and substitute for μ, p, q into (8) and (9). Substituting for α and β into (10) we get a quadratic for r, and the condition for this to have real roots becomes, after considerable reduction,

$$\Delta\mu > 0. \quad \ldots\ldots(12)$$

In the case of the hyperbola, the two values of μ have opposite signs, so that we must choose the value which has the same sign as Δ. In the case of the ellipse, if the equation is arranged so that $a > 0$, then $\Delta < 0$ (for if not S would be positive for all x and y), and both values of μ are negative. Hence (12) is satisfied for both values of μ.

So, for this ntethod, the rules are :

(a) for a hyperbola, choose the value of μ which has the same sign as Δ.

(b) for an ellipse, choose the value of μ which makes $\mu a + 1 > 0$.

4. Since the roots of (1) are Δ times the roots of (11), it appears that there should be a connection between the rules for the two methods. However, the rules are very different as given above : method 1 imposes one condition on λ which is the same for all conics, whereas method 2 imposes different conditions on μ according as the conic is a hyperbola or an ellipse. In order to show the connection, we derive equivalent conditions for the two methods.

Method 1. Adding (2) and (3), we get

$$-\Delta(a+b) - 2C\lambda > 0.$$

On using (1), this gives

$$\Delta^2/\lambda - C\lambda > 0. \quad \ldots\ldots(13)$$

Method 2. Adding (5) and (7) we get

$$\mu(a+b)+2>0.$$

On using (11), this gives

$$1-C\mu^2>0. \quad \text{.....(14)}$$

Together, (12) and (14) give

$$\Delta(1/\mu - C\mu)>0, \quad \text{.....(15)}$$

which is the same condition as (13), with $\lambda=\Delta\mu$.

Now, if (15) is true, so are (12) and (14). For, in the case of a hyperbola, $C<0$, so that (14) is satisfied for both values of μ, and (15) implies (12). In the case of an ellipse, we have seen that (12) is satisfied for both values of μ; and (15) implies (14).

5. In conclusion, I should point out that method 1 seems much the best for finding the foci. Method 2 is rather lengthy, and the methods involving the circular points lead to the solution of a quartic equation (with two real and two imaginary roots), which may be difficult to solve in practice.

E. J. F. PRIMROSE.

2289. *Quartics with values which are perfect squares.*

If $f(x)$ is a polynomial (with rational coefficients) in x of the fourth degree, and one rational value x_0 of x is known for which $f(x_0)$ is a square, then methods are known by which, in general, further rational values x_r can be found so that $f(x_r)$ is a square. The purpose of this note is to give a method by which x_r is determined by a recurrence relation. In what follows all letters represent rational numbers.

If it is possible to express $f(x)$ in the form, which we call discriminant form $(vx^2+qx+m)^2-4(ux^2+px+l)(wx^2+rx+n)$, and one value of x, say x_0, is known for which $f(x)$ is a square, it then follows that the equation

$$F(y,\,x_0)=y^2(ux_0^2+px_0+l)+y(vx_0^2+qx_0+m)+(wx_0^2+rx_0+n)=0$$

has rational roots say y_1 and y_{-1}.

The equation $F(y_1,x)=x^2(uy_1^2+vy_1+w)+x(py_1^2+qy_1+r)+(ly_1^2+my_1+n)=0$ is known to have one rational root x_0 and therefore it has in general a second rational root x_1.

It follows that as the equation $F(y,\,x_1)=0$ has one rational root y_1, it has a second rational root y_2 and the equation $F(y_2,\,x)=0$ has rational roots x_1 and x_2.

Continuing thus we can obtain a sequence of numbers, $x_0,\ y_1,\ x_1,\ y_2,\ x_2,$ such that $F(y,\,x_r)=0$ has rational roots $y_r,\ y_{r+1}$ and so the discriminant $f(x_r)$, is a square.

By taking y_{-1} instead of y_1 we may extend the sequence to the left to get $\ldots,\ x_{-2},\ y_{-2},\ x_{-1},\ y_{-1},\ x_0,\ y_1,\ x_1,\ y_2,\ x_2,\ \ldots$

This gives a method of obtaining further solutions x_r of the equation $f(x)=$square when one, x_0, is known.

We shall now show that if $f(x_0)=g^2$ it is possible to express $f(x)$ in the form

$$(vx^2+qx+m)^2-4(ux^2+px+l)(wx^2+vx+m).$$

Put $x=z+x_0$. Then $f(x)=az^4+bz^3+cz^2+dz+g^2$

$$=(g+dz/2g)^2+z^2(c-d^2/4g^2+bz+az^2)$$

$$=(vx^2+qx+m)^2-4(ux^2+px+l)(wx^2+vx+n)$$

where $v=0$ and $p^2=4ul$.

This shows that it is possible to express $f(x)$ in discriminant form but we should note that if it is done in this way $ux_0^2+px_0+l=0$, the equation from $F(y,\,x_0)=0$ has only one root and the sequence extends only one way from x_0. In practice it is often easy to express $f(x)$ in discriminant form by inspection

The process of calculating successive x_r may be shortened by noticing that

$$x_{r-1}x_r = (ly_r^2 + my_r + n)/(uy_r^2 + vy_r + w),$$

$$x_r x_{r+1} = (ly_{r+1}^2 + my_{r+1} + n)/(uy_{r+1}^2 + vy_{r+1} + w)$$

and hence $x_{r-1}x_r^2x_{r+1}$ and $x_r(x_{r-1} + x_{r+1})$ are symmetric functions of y_r and y_{r+1} the roots of $F(y, x) = 0$ and can be expressed in terms of x_r. From this we find after some heavy algebra that x_{r-1} and x_{r+1} are the roots of

$$x^2(ax_r^2 + hx_r + g) + x(hx_r^2 + bx_r + f) + gx_r^2 + fx_r + c = 0,$$

where $a = M^2 - LN$, $b = 2(Q^2 - PR) - (UN - 2VM + WL)$, $c = V^2 - UW$, $f = UR - 2VQ + WP$, $g = Q^2 - PR$, $h = PN - 2QM + RL$, the capital letters being the cofactors in

$$\Delta = \begin{vmatrix} u & v & w \\ p & q & r \\ l & m & n \end{vmatrix}.$$

Hence successive x_r may be calculated from whichever is more convenient of the recurrence relations.

$$x_{r+1} + x_{r-1} = -(hx_r^2 + bx_r + f)/(ax_r^2 + hx_r + g),$$

$$x_{r+1}x_{r-1} = (gx_r^2 + fx_r + c)/(ax_r^2 + hx_r + g).$$

It may be verified that $(vx^2 + qx + m)^2 - 4(ux^2 + px + l)(wx^2 + rx + n) \equiv \Delta^{-2}[(hx^2 + bx + f)^2 - 4(ax^2 + hx + g)(gx^2 + fx + c)]$.

Illustration I.

Find positive integral x, y so that $x + y$ and $x^2 + y^2$ are each cubes. Put $x + iy = (a + ib)^3$, *i.e.* $x = a^3 - 3ab^2$, $y = 3a^2b - b^3$; then $x^2 + y^2 = (a^2 + b^2)^3$ and it still remains to make $x + y =$ cube.

$$x + y = a^3 + 3a^2b - 3ab^2 - b^3 = (a + kb)^3, \text{ say.}$$

Then $$3a^2(k - 1) + 3ab(k^2 + 1) + b^2(k^3 + 1) = 0, \quad \text{.....(i)}$$

and we must have $$9(k^2 + 1)^2 - 12(k - 1)(k^3 + 1) = \text{square.} \quad \text{.....(ii)}$$

The obvious solutions $k = -1$, $k = +1$ give $a, b = 1, 1$ and $-1, 3$ and these give $x, y = -2, 2$ and $18, -26$. As these are not both positive we must find further values of k satisfying (ii)

This we write in discriminant form

$$(3k^2 + 3)^2 - 4(3k^2 - 3)(k^2 - k + 1) = \text{square}$$

From $u = 3$	$v = 3$	$w = 1$,	we get $U = 3$	$V = 3$	$W = 0$
$p = 0$	$q = 0$	$r = -1$		$P = 0$	$Q = 6$	$R = -18$
$l = -3$	$m = 3$	$n = 1$		$L = -3$	$M = 3$	$N = 0$.

Dividing all cofactors by 3, we have

$$a = 1, \quad b = 10, \quad c = 1, \quad f = -10, \quad g = 4, \quad h = 2.$$

Hence if k satisfies (ii) then the roots of

$$x^2(k^2 + 2k + 4) + x(2k^2 + 10k - 10) + 4k^2 - 10k + 1 = 0 \text{ also satisfy (ii).}$$

Put $k = -1$, $3x^2 - 18x + 15 = 0$; $k_{-1} = 5$, $k_1 = 1$.

$k_1 = 1$, $7x^2 + 2x - 5 = 0$; $k_0 = -1$, $k_2 = \frac{5}{7}$.

Obtaining further values from the recurrence relation

$$k_{n+1}k_{n-1} = (4k_n^2 - 10k_n + 1)/(k_n^2 + 2k_n + 4) \quad \text{.....(iii)}$$

we get a sequence $\ldots \frac{707}{313}, -\frac{17}{13}, 5, -1, 1, \frac{5}{7}, -\frac{401}{191} \ldots$

$k=5$ gives a, $b=3$, -1 or 7, -2 which in turn give x, $y=18$, -26 and 259, -286. $k=\frac{5}{7}$ gives a, $b=2$, -7 or 39, 7 and the latter value gives two positive integral values for x, y viz. 53586, 31598.

Illustration II.

Solve $tx^4+ux^2+1=y^2$ where $t\neq$ square .....(i)

If tx^4+ux^2+1 can be put in discriminant form, we have shown that it can be put in the closer form

$$(hx^2+bx+f)^2-4(ax^2+hx+g)(gx^2+fx+c).$$

Equating coefficients of x^3 and x to zero, we have

$$2h(b-2g)=4af$$

and $$f(b-2g)=2ch$$

From these it follows that $ch^2=af^2$ .....(ii)

Comparing the coefficients of x^4 and x^0,

$$h^2-4ag=t(f^2-4gc)$$

which with (ii) gives $af^2-4agc=ct(f^2-4gc)$.

Hence $a=ct$,

and substitution in (ii) gives $h^2=tf^2$.

Since $t\neq$ square it follows that $h=f=0$.

From equating the coefficients of x^2 and x^0 we have

$$b^2=4g^2+4tc^2+u$$

and $$-4gc=1.$$

Hence $b^2=4(-1/4c)^2+4tc^2+u$

i.e. $16tc^4+4uc^2+1=4c^2b^2$.

Thus $2c=x_1$, $2cb=y_1$, where x_1, y_1 is a solution of (i) and $a=tx_1/2$, $b=y_1/x_1$, $c=x_1/2$, $f=0$, $g=-1/2x_1$, $h=0$.

So if x_1, x_r are solutions of (i), then so are the roots of

$$x^2(tx_1^2x_r^2-1)+2xy_1x_r+(x_1^2-x_r^2)=0.$$

Putting $x_r=x_1$ and x_2 we can state this result in other words. If x_1 and x_2 are solutions of $tx^4+ux^2+1=y^2$, then so are

$$2x_1y_1/(tx_1^4-1) \quad \text{and} \quad (x_1y_2\pm x_2y_1)/(tx_1^2x_2^2-1).$$

These results are true when $t=$ square as well as when $t\neq$ square.

In the same way it may be shown that if x_1 and x_2 are solutions of

$$x^3+k=sy^2,$$

then so are $x_1(x_1^3-8k)/4sy_1^2$ and $\{x_1x_2(x_1+x_2)+2k\pm 2sy_1y_2\}/(x_1-x_2)^2$.

R. C. L.

2290. *A remark on torsion.*

The Saint-Venant theory of the uniform torsion of cylindrical bars is based (in the standard textbooks) on a system of displacements guessed from analogy with the special case of a circular cylinder. A somewhat different approach is given here which has the advantage of introducing the stress and displacement functions more symmetrically.

We will take the z-axis parallel to the generators of the cylinder and seek a stress system in which the only non-zero strains are e_{xz}, e_{yz}. From the isotropic stress-strain law only the stresses $\widehat{xz} = \mu e_{xz}$, $\widehat{yz} = \mu e_{yz}$ do not vanish. [The strains are expressed in terms of displacements by

$$e_{xx} = \frac{\partial u}{\partial x}, \; e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.]$$

The equations of equilibrium reduce to

$$\frac{\partial \widehat{xz}}{\partial z} = \frac{\partial \widehat{yz}}{\partial z} = 0, \quad \frac{\partial \widehat{xz}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} = 0.$$

Thus e_{xz}, e_{yz} are independent of z and

$$\frac{\partial e_{xz}}{\partial x} = -\frac{\partial e_{yz}}{\partial y}. \qquad \text{(1)}$$

The equations of compatibility (Love : *Mathematical theory of elasticity*, p. 49) then simplify to

$$\frac{\partial}{\partial x}\left(-\frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{yz}}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{yz}}{\partial x}\right) = 0.$$

Therefore

$$-\frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{yz}}{\partial x} = 2\alpha, \qquad \text{(2)}$$

where α is a constant whose physical meaning will appear later.

It is convenient to rewrite (1) and (2) as

$$\frac{\partial (e_{xz} + \alpha y)}{\partial x} = -\frac{\partial (e_{yz} - \alpha x)}{\partial y}. \qquad \text{(3)}$$

$$\frac{\partial (e_{xz} + \alpha y)}{\partial y} = \frac{\partial (e_{yz} - \alpha x)}{\partial x}. \qquad \text{(4)}$$

Now (3) is satisfied automatically on writing

$$e_{xz} + \alpha y = \alpha \frac{\partial \psi}{\partial y}, \quad e_{yz} - \alpha x = -\alpha \frac{\partial \psi}{\partial x}, \qquad \text{(5)}$$

where ψ is a function of x, y alone.

Substituting in (4), ψ is seen to satisfy

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \qquad \text{(6)}$$

Similarly (4) is satisfied automatically by

$$e_{xz} + \alpha y = \alpha \frac{\partial \phi}{\partial x}, \quad e_{yz} - \alpha x = \alpha \frac{\partial \phi}{\partial y}, \qquad \text{(7)}$$

where ϕ is a function of x, y alone, and from (3)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \qquad \text{(8)}$$

Comparing (5) and (7)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad \text{(9)}$$

the familiar Cauchy-Riemann equations of function theory. The harmonic functions ϕ, ψ are known, respectively, as the displacement and stress functions.

The displacements are easily expressed in terms of ϕ. The equations

$$e_{xx}=\frac{\partial u}{\partial x}=0, \quad e_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0, \quad e_{yy}=\frac{\partial v}{\partial y}=0. \qquad \text{.....(10)}$$

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=-\alpha y+\alpha\frac{\partial \phi}{\partial x}, \quad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=\alpha x+\alpha\frac{\partial \phi}{\partial y} \qquad \text{.....(7)}$$

have the obvious solution

$$u=-\alpha yz, \quad v=\alpha xz, \quad w=\alpha\phi(x, y). \qquad \text{.....(11)}$$

From (11) we can interpret α as the angle of twist per unit length of the bar.

The boundary condition of the Saint-Venant theory is that the curved surface shall be free from stress. We take as coordinate on the surface the arc length s measured in a counter-clockwise direction from a fixed generator along a line of curvature. Let ν be the outward normal to this curve.

The boundary condition then becomes

$$\widehat{\nu z}=\widehat{xz}\cos(\nu, x)+\widehat{yz}\cos(\nu, y)=0$$

i.e.

$$\widehat{xz}\frac{\partial y}{\partial s}-\widehat{yz}\frac{\partial x}{\partial s}=0 \qquad \text{.....(12)}$$

since

$$\frac{\partial x}{\partial s}=\cos(s, x)=-\cos(\nu, y)$$

$$\frac{\partial y}{\partial s}=\cos(s, y)=\cos(\nu, x).$$

The condition (12) can easily be expressed in terms of ϕ by means of (7). However, it becomes much simpler in terms of the stress function ψ. From (5)

$$e_{xz}=\alpha\frac{\partial}{\partial y}[\psi-\tfrac{1}{2}(x^2+y^2)], \quad e_{yz}=-\alpha\frac{\partial}{\partial x}[\psi-\tfrac{1}{2}(x^2+y^2)],$$

and so

$$\widehat{\nu z}=\mu\alpha\frac{\partial}{\partial s}[\psi-\tfrac{1}{2}(x^2+y^2)]. \qquad \text{.....(13)}$$

Thus (12) becomes

$$\psi-\tfrac{1}{2}(x^2+y^2)=\text{const.} \qquad \text{.....(14)}$$

on the boundary. The simplicity of (14) and (13) (which obviously also holds on any curve inside the body) is the reason for the importance of the stress function.

R. C. T. SMITH.

2291. *Note on reversed numbers.*

In connection with Notes 1958 and 2065 on the reversion of numbers, we have an analogous property for the sum of two such numbers if we write the properties in their direct forms, namely

$$\overline{ab}-\overline{ba}=(a-b)(A^k-1)$$

$$\overline{ab^2}-\overline{ba^2}=(a^2-b^2)(A^{2k}-1)$$

where k is the number of digits in each group a and b and A is the number base. Then the corresponding property is

$$\overline{ab}+\overline{ba}=(a+b)(A^k+1).$$

For example,

$$32+23=5\,.\,11,$$

$$2137+3721=58\,.\,101.$$

A. BUCKLEY.

REVIEWS.

History of Theories of the Aether and Electricity. I. By SIR EDMUND WHITTAKER. Pp. xiv, 434. 32s. 6d. 1951. (Nelson)

" First published 1910. Revised and enlarged edition 1951." So one reads on the back of the title page. What makes this almost certainly a record is that the revision and enlargement have been carried out by the original author of forty-one years ago.

Every reader of the *Gazette* knows Sir Edmund Whittaker to be one of the pre-eminent mathematicians of the times. Every reader must be aware, too, of the immense range of Sir Edmund's original contributions to mathematics and mathematical physics. Also, as the review-pages of this journal bear witness, Sir Edmund has in recent years become increasingly recognised as an authority upon, and a brilliant exponent of, the general historical and philosophical aspects of mathematical physics. But there are great achievements to his credit with which not every reader may be well-acquainted. One cannot forbear mentioning here that it is he to whom we owe, for instance, the general introduction into this country of the teaching of the theory of functions of a complex variable. It was he too who in his *Analytical Dynamics* (1904 and later editions) made generally known the dynamical theories which have since played such a great part in the development of quantum mechanics (though at the time he was criticised for giving prominence to such academic aspects of the subject!). And not every reader may realise that so many years ago Sir Edmund established himself as the leading historical scholar in the field with which the present book is concerned.

In mitigation of any such lack of awareness, it may be pleaded that the original work has been out-of-print for something like thirty years. But this the author has long intended to remedy. The reason for his delay in doing so, as he explains in the Preface of the new edition, has been his view " that any new issue should describe the origins of relativity and quantum theory, and their development since 1900 ". It was not until his retirement from his Edinburgh professorship, which itself was postponed on account of the war, that he was able to take up this formidable project. His plan is to produce the work in two volumes, the present one dealing with the " classical " theories and a second bringing the history up to the present time.

The original book gave a superbly well-knit account of its subject. It is gratifying to see that in the new edition Sir Edmund has been able to retain the bulk of this account in its original form. So skilfully has this been done that the extreme care with which the revision has been carried through may easily be grossly underestimated. It is, of course, evident that Chapter I has been largely re-written, that Chapter XII is entirely new, and that Chapter XIII (formerly XII) contains much new material while it omits those parts of the original which are appropriately held over for the second volume. But throughout the other chapters there are many fresh paragraphs and many extensions to the footnotes and references. The clearest evidence of the thoroughness of the revision is furnished by the " Index of Authors Cited ": impressive as this was in the first edition, it is now considerably increased in both the number of names and the number of references to each.

The full title of the 1910 edition was *A History of the Theory of the Aether and Electricity, from the age of Descartes to the close of the nineteenth century.* This is still an accurate description. In the Preface, presumably with his second volume in mind, the author justifies his retention of the word " aether " in the title of the whole work. But no such special justification is required so far as the present volume is concerned. A big part of the scientific work of the period did, as a matter of history, consist in theories of what the workers themselves

called the aether. What was not known to these workers until about the last fifty years of the period was that their theories of the aether as the " luminiferous medium " and their theories of electric and magnetic phenomena were destined to become so intimately related. Consequently, throughout most of the book, chapters or series of chapters have to deal alternately with these two aspects.

After a brief sketch of primitive physics, Chapter I shows that the story begins effectively with Descartes who " was the first to bring the aether into science ". It provides an account of his work followed by that on theories of light by Fermat, Hooke, Huygens, Newton and their contemporaries. Chapter II traces the history of electric and magnetic science, essentially electrostatics and magnetostatics, from the earliest times up to Green's introduction of the (scalar) potentials. Chapter III traces the almost contemporaneous but largely independent development of " Galvinism " or current-electricity as far as the work of Ohm on conductivity and including the laying of the foundations of electrodynamics, chiefly by Ampère.

Chapter IV resumes the history of the " luminiferous medium ", taking it from Bradley to Fresnel, that is, from the time when the corpuscular theory of light was predominant to that when the wave-theory became firmly established by Young and Fresnel. The success of these two workers was due fundamentally to their recognition of the transverse character of light waves. This led at once to the view that the aether behaves like an elastic solid. The next chapter deals with " the problem of devising a medium whose vibrations, calculated in accordance with the correct laws of dynamics, should have the same properties as the vibrations of light ". Attempts to solve this problem are associated with the names of nearly all the great mathematical physicists of the nineteenth century, including Poisson, Cauchy (whom it is now surprising to recall in this guise), Green, MacCullagh, W. Thomson (Lord Kelvin), Stokes and Boussinesq. The central problem was solved by MacCullagh in 1839. Also a great deal of progress was made in at any rate the formal mathematical representation of the interaction of light and matter, culminating in Boussinesq's demonstration that a single set of equations served to describe the motion of the aether in all space, irrespective of the presence of material bodies, while another set could be formulated to describe the interaction of the aether, and whatever particular kind of matter was present.

All this work had nothing explicitly to do with electromagnetism. Work was started somewhat later, after the discovery of the equations of electromagnetic fields, on the invention of mechanical models of the aether which would reproduce those equations when an appropriate correspondence was established between their mechanical properties and the field-parameters. These models, associated principally with the names of Maxwell, W. Thomson, MacCullagh, Riemann, Fitzgerald and Helmholtz, are described in due course in Chapter IX. They are generally regarded as typifying the outlook of nineteenth-century physicists. The reviewer's fragmentary reading of Maxwell's writings seems to indicate, however, that at any rate Maxwell himself tended to regard such models rather more as means of suggesting possible mathematical relations for the electromagnetic field than as providing mechanical " explanations " of those relations. Sir Edmund Whittaker's comprehensive account of Maxwell's researches may not altogether bear out this impression ; but it does seem to admit the interpretation that Maxwell was somewhat more non-committal in his attitude towards models of the aether than some of the other investigators.

The intervening chapters describe the discoveries of Faraday, the mathematical researches of Maxwell's predecessors, and the work of Maxwell himself on the electromagnetic field. Then Chapter X describes the work of the

followers of Maxwell in which the general theory of electromagnetism attained practically the form it still possesses.

Most of the remaining work described concerns in one way or another the atomicity of matter and of electric charge. Chapter XI deals with experimental discoveries after Faraday up to J. J. Thomson's discovery of the electron. Chapter XII on " Classical Radiation-theory " is largely a description of the empirical work on spectral series, and the empirical and theoretical work on black-body radiation, that later led to the discovery of quantum phenomena. The final chapter centres round the Lorentz electron-theory of optical phenomena and of electric and thermal conduction.

The book is a history of *theories*. But the author supplies very clear statements of the experimental discoveries at all stages. The book is about the aether and electricity, that is to say, about the most fundamental parts of physical theory, though quite how fundamental they are was becoming more fully appreciated towards the end of the period here dealt with and was to become even more evident during the period to be treated in the second volume. But the author keeps the reader carefully informed about relevant progress in other parts of physical theory, as for instance in elasticity and in thermodynamics, though without allowing him ever to lose the main threads of the story.

Sir Edmund's profound scholarly treatment gives us a convincing account of the general physical concepts underlying, and resulting from, each stage in the evolution of the theories. He does more, however, than merely write about the theories ; he actually re-derives in outline their main mathematical developments. This is perhaps the foremost among the many shining virtues of the work. For it enables the author to show in the clearest and most concise manner possible both the way in which the advances were made and also the precise relations between the contributions of the many workers engaged.

In his Eddington Memorial Lecture this year, Sir Edmund Whittaker has himself said, " The fame of a theoretical physicist rests on the part that his ideas have played in the history of science : it does not necessarily detract from his importance if none of them survive into the physics of his remote successors." It is in this spirit that he has written the history of this part of science. He gives a faithful account of the whole of its development, not neglecting those workers whose ideas played any significant part even if their ideas did not survive even their own lifetimes.

" There is this curious quality about the really great discoveries of science, that they carry general conviction, they need only to be expressed to be accepted. I realise of course that scientific controversies are not unknown, that there are initial stages in which violently different views are held and maintained with great heat. We must recognize too that scientific views are subject to continuous modification and restatement as knowledge advances. But I think that anyone who has studied science will understand what I mean. The discoveries of the giants of the past are built upon, not discarded. They form a solid foundation to a superstructure which is added by their successors." This is not a quotation from Sir Edmund Whittaker. It was said by Sir Lawrence Bragg in his Radford Mather Lecture in 1950 (*Advancement of Science* (1950), 279–84). But it seems to the reviewer to be singularly apt as a commentary upon the material in this volume and as providing an approach to the material to be given in the succeeding volume. Different as the developments there to be described will appear, they will rest upon the foundations so well depicted in the present volume.

There is, however, a more subtle point. We can agree with Sir Lawrence Bragg despite the fact that the discoveries to which he refers have to be expressed in terms of basic concepts—aether, time, space, mass, electrons,

and so on—that are themselves subject to drastic modification. Indeed, even apart from what he says, we recognize these discoveries as marking what we must call progress for the simple reason that they give us increased ability to predict future experience. Why this should be so in spite of the impermanence of the basic concepts is a very deep problem.

Throughout the period covered by the present book, the basic concepts of time, space, mass and force remained effectively unchanged. The further progress in the ensuing period has depended on calling these concepts in question. We look forward with eagerness to Sir Edmund Whittaker's account of this phase and also of the views of himself and other thinkers upon such deep problems as the one just mentioned.

Out of the riches of his mathematical and historical scholarship, Sir Edmund Whittaker has given us a very great book. It is fitting that a review should include an expression of gratitude for his readiness to make generally accessible the fruits of his unique intellectual experience as well as an expression of good wishes for the completion of this high undertaking. W. H. McCrea.

An Introduction to Applied Mathematics. By J. C. Jaeger. Pp. xiii, 445. 35s. 1951. (Oxford University Press)

By the title of his book, Professor Jaeger means an introduction to the techniques and functions required for the solution of problems which arise in physics and engineering. In the main this consists of an account of differential equations applied to physical problems. It is suggested in the preface that such a course will prove more useful than the normal practice of developing skill in the solution of comparatively difficult problems in statics and dynamics.

After an introductory chapter, the ordinary linear differential equation with constant coefficients is treated in Chap. II, the use of the operator D being followed by a short account of the method of Laplace transforms. Chap. III deals with equations of the first order. In Chaps. IV and V the results and methods so far obtained are applied first to mechanical problems of free, forced and damped vibrations, including the normal modes of systems with several degrees of freedom, and then to electric circuit theory. The latter chapter points the analogy between electrical and mechanical systems. There is a brief introduction to servo-mechanisms ; the triode circuit affords an opportunity of suggesting a method of approximate solution of a non-linear equation. Chap. VI, on vectors, is a preliminary to Chaps. VII and VIII, on particle and rigid dynamics. Orbital motions include those of an electron in electric and magnetic fields and there is a further investigation of non-linear systems. In these chapters, the solutions are based on the equations of motion. It is, perhaps, to be regretted that the principles of conservation of linear momentum, angular momentum and energy are not established and used ; in the section on the gyrostat there is a forward reference to the energy equation, the use of which is delayed to Chap. IX, where Lagrange's equations are obtained for independent generalised coordinates. Chap. X, entitled " Boundary Value Problems ", is devoted to deflection of beams, Chap. XI to Fourier's series and integrals. In Chap. XII the solution in series of Bessel's and Legendre's equations leads to the chief properties of the Bessel and gamma functions and the Legendre polynomials ; there is also an introduction to the Green's function and Mathieu's equation. Chap. XIII shows how three important partial differential equations arise in physics, solutions being obtained in Fourier series and otherwise. The final chapter deals with numerical methods. Finite differences are introdueed and used for the numerical solution of differential equations. There is a simple worked example to illustrate the use of relaxation methods.

The three constituents of lucid exposition, suitable illustrative examples

and good sets of examples for the student, go to make this a very good text-book, which will be welcomed by all who are concerned with the teaching of mathematics for degrees in science and engineering. C. G. P.

Conduction of Heat in Solids. By H. S. CARSLAW and J. C. JAEGER. Pp. viii, 386. 30s. 1947. (Oxford, at the Clarendon Press)

When Carslaw's *Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat* was published in 1906, British mathematicians were just beginning to realise that the infinite series and infinite integrals arising in mathematical physics ought to be and could be handled in a rigorous manner. In earlier days the standard had been that of Thomson and Tait's *Natural Philosophy*, where can be found a " proof " of the statement, now known to be false, that the Fourier series of a continuous function always converges to the function.

Carslaw's book began with the theory of real numbers, treated by Dedekind's method, and went on to the general theory of infinite series and integrals, as a prelude to a rigorous treatment of Fourier series and Fourier integrals, The second half of the book was devoted to the mathematical theory of the conduction of heat in solids ; here the problems discussed by Fourier in his *Théorie analytique de la chaleur* were examined afresh, and the solutions were obtained by methods satisfying the demands of mathematical rigour ; many other problems of the theory were treated. Chapters were also devoted to the method of images and to the use of Green's functions for solving heat conduction problems. The book played an important part in the popularisation of a correct treatment of infinite series and integrals in the problems of mathematical physics.

When a new edition became necessary the book was split into two halves, entitled respectively " Fourier's Series and Integrals " and " Introduction to the Mathematical Theory of the Conduction of Heat in Solids " ; these both appeared in 1921. The latter was a revised and considerably enlarged version of the second half of the 1906 book, and contained an important new feature : a chapter on the use of contour integrals in heat conduction problems, which was in effect a discussion of the application of the operational calculus, treated by Bromwich's method.

The present book is based on that of 1921, but has been completely re-written in collaboration with Professor Jaeger. Nevertheless, it still bears in its structure the impress of its history ; the problems of heat conduction theory are classified according to the mathematical tools required to solve them, each tool in turn being discussed and then applied to a multitude of particular problems. The chief new feature of the book is the much greater part played by the operational calculus, now treated by the Laplace transformation instead of by the Bromwich integral.

The book may be described as a compendium of methods and results relating to the solution of the heat conduction equation for special types of boundary conditions. After chapters dealing with one-dimensional problems, including problems on linear flow with lateral radiation, there follow chapters on problems that are handled most naturally in rectangular, in cylindrical, and finally in spherical polar coordinates. The dominant technique in these chapters is the method of separation of variables, leading to the use of ordinary and generalised trigonometric series, of Fourier integrals, of multiple Fourier series, and of series involving Bessel functions and spherical harmonics. In the next chapter the use of sources and sinks leads to solutions in the form of definite integrals and to the method of images. The authors then proceed to describe the applications of the Laplace transformation, and in particular its usefulness in obtaining the behaviour of a system for small and for large values

of the time parameter. The last three chapters contain a discussion of the use of Green's functions, further applications of the Laplace transformation, and a description of the use of potential theory methods in steady temperature problems.

It is perhaps to be regretted that the authors have not found space for some treatment of the properties of the heat conduction equation under general boundary and initial conditions. As an equation of parabolic type, its properties differ in certain essential ways, for instance, in the character of the physically appropriate types of boundary conditions, from those of the potential equation (elliptic type) on the one hand and the wave equation (hyperbolic type) on the other. Some discussion of this general theory might have been helpful to readers who are concerned with boundary and initial conditions that do not fit into any of the special categories discussed in this book, as distinct from those who can compel their experimental set-up to conform closely with a problem that has already been solved, as can often be done in experiments concerned with the determination of physical constants. The authors themselves remark that the uniqueness theory of the equation is not yet complete ; further investigations in this field might well be rewarding. It is also surprising that the authors have not mentioned that the heat conduction equation is also the diffusion equation ; diffusion problems do not appear to be mentioned at all.

It may be that these omissions are a natural consequence of the past history of the book. Certainly, within the limits that the authors have set themselves, they have done a very thorough job, and there must be few known results for special types of boundary conditions that have slipped through their net.

F. SMITHIES.

Existence Theorems in Partial Differential Equations. By DOROTHY L. BERNSTEIN. Pp. vii, 228. 16s. 1950. Annals of Mathematics Studies, 23. (Princeton University Press ; Geoffrey Cumberlege, Oxford University Press)

Professor Bernstein has compiled a comprehensive, and most interesting, collection of existence theorems, which will be a welcome addition to the libraries of those who are interested in the theory of partial differential equations. All conditions of continuity, differentiability, etc., which are assumed in each theorem are meticulously stated ; and the proofs, which are mostly given in outline only, are nevertheless thorough and rigorous. Ample references are provided throughout the text to literature dealing in detail with each problem, and the book thus fulfils a valuable function in helping the reader to find both statements and proofs of the less well-known theorems. Unfortunately the very abridged and rather artificial notations employed, though providing excellent economy of words in the careful statements of conditions, prevent the work from being really pleasant or easy reading.

There is one serious omission. No comprehensive index or classification is given of the many existence theorems themselves, so that it is impossible, without reading through whole chapters, to pick out quickly the theorem which may apply to any given problem. For this reason, the book will not appeal to those for whom, according to the preface, it is primarily intended : namely the engineers and applied mathematicians whose main interest is in obtaining numerical solutions of given equations.

D. H. P.

The Topology of Fibre Bundles. By N. E. STEENROD. Pp. viii, 224. 32s. 6d. 1951. Princeton Mathematical series, 14. (Princeton University Press ; Geoffrey Cumberlege, London)

A fibre-space B over a base-space X with fibre Y generalises the notion of a topological product $X \times Y$. A projection $p : B \to X$ is given and the counter-

image of each point of X under p is homeomorphic to Y; moreover, with respect to a certain system of neighbourhoods in X, B is " locally " homeomorphic to a topological product. Thus, for example, a covering space over X is a fibre-space over X, the fibre being, in this case, discrete. A fibre-space becomes a fibre bundle when there is given, as additional structure, a topological group of homeomorphisms of Y which serves, as it were, to show how the fibres are bound together to form the fibre-bundle.

After giving some simple but illustrative examples of fibre-bundles (the Möbius band over the circle, with the line as fibre; a Lie group over a coset space with closed subgroup as fibre; the set of tangent vectors at points of a differentiable manifold, with a representative tangent plane as fibre), the author devotes the rest of Part I to a treatment of the general properties of fibre-bundles, which is valuable, *inter alia*, for indicating the way in which the study of tensor fields on manifolds has given rise to and influenced developments in bundle-theory. Thus, for example, considerable attention is focussed on the question of whether the fibre-bundle admits a section, that is, a map $f : X \to B$ such that the composite map $pf : X \to X$ is the identity. Such a section would correspond, in the case in which B is the set of tangent vectors over the manifold X, to the existence of a continuous vector field over X.

In Part II, the homotopy theory of bundles is given. This part opens with a survey of the Hurewicz homotopy groups which can be recommended to any student of topology irrespective of his speciality. The fundamental " lifting homotopy " theorem is then proved, and applied, together with the Freudenthal suspension theorems, to the calculation of the homotopy groups of spheres. This leads to a description of the work of Eckmann and G. W. Whitehead on the homotopy groups of the orthogonal groups and unitary groups and the Stiefel manifolds (systems of orthogonal vectors over euclidean n-space).

In Part III, the base-space is taken to be a cell-complex. Then a partial section is a map, g, of a closed sub-complex X_0 into B such that $pg : X_0 \to X_0$ is the identity map. The author shows how the problem of extending a partial section leads to a certain obstruction cocycle with coefficients in a homotopy group of the fibre, and then describes the necessary cohomology theory of a complex (modulo a sub-complex) with local coefficients which he introduced in 1943. The theory is then developed to yield generalisations of the Hopf classification and extension theorems for maps of a complex into a sphere, and further more specialised applications are made to the differential geometry of manifolds and to the study (due to Whitney) of sphere-bundles (fibre-bundles in which the fibres are spheres and the group is the orthogonal group).

This book must be strongly recommended to all specialists in the field and should be of great interest and value to others whose work lies on the fringe of algebraic topology or differential geometry. In the lucidity of its treatment and its scope, it achieves the high standard to be expected from one who has made so many important contributions to the development of its subject.

P. J. HILTON.

Observing the Heavens. By PETER HOOD. Pp. 64. 5s. 6d. net. 1951. (Oxford University Press)

This attractive book is the first of a new series of annotated picture books for children of 11 to 15 on subjects connected with physical or natural science, though it seems more suitable for rather older children.

It starts with chapters on the use and history of astronomy, the use and making of a simple telescope. It then goes on to observation of the moon, the sun, the stars (including maps of the heavens), comets, double stars and nebulae with a final chapter on the galaxy and other galaxies. This is fol-

lowed by lists of about 40 constellations and 20 stars and favourable dates up to 1970 for observing the planets.

In some of the wide margins the more important constellations might have been shown in simple diagrams, like those used by the Air Training Corps; these would be useful in learning one's way about the sky.

The book is profusely illustrated and the maps of the stars are very good. Altogether a stimulating and ambitious book, and remarkably cheap.

A. W. SIDDONS.

Theoretische Mechanik. By G. HAMEL. Pp. xvi, 796. DM 63: bound, DM 66. 1949. Die Grundlehren der Mathematischen Wissenschaften, Band LVII. (Springer, Berlin)

This book is a welcome and important addition to the well-known "yellow back" series of mathematical texts. In the opinion of the writer, it goes far to show that it is still possible to write in an interesting and stimulating manner on a topic already rich in its literature.*

It is not an elementary treatise and the author may well intend it to be a sequel to his *Elementare Mechanik*. Based on the author's lectures delivered at the Technischen Hockschule, Berlin, it displays a thoroughness in all its details which cannot but be admired, be it in the text or in the worked examples, of which there are nearly two hundred, mostly collected at the end of the book. There are also a number of welcome historical notes throughout the text.

The book deals with the dynamics of a particle, the statics of rigid and elastic bodies, analytical mechanics and the dynamics of rigid bodies in this order. Some may urge that the insertion of the analytical portions before the discussion of rigid dynamics breaks the continuity of the book. But there is a definite advantage to be gained in introducing the analytical methods in Mechanics at an early stage.

The first part of the book is divided into nine chapters, the first on the dynamics of a particle being of an introductory nature. The motion of a particle in a resisting medium is discussed at length. There is a historical critique of the principles of mechanics and the chapter closes with an introduction to Special Relativity.

The chapters on statics deal with systems of a finite number of degrees of freedom, with the theory of elasticity, and with the hydrodynamics of incompressible fluids. There is an invaluable discussion of the principle of virtual work from its early beginning to its final formulation by Lagrange.

The sections on the statics of chains, elastic shells and plates and the elements of the theory of elasticity are very elegant in their presentation.

The discussion of finite and infinitesimal displacements of a rigid body is one of the most complete yet given in a text-book on mechanics, and the part dealing with Rodriguez's formulae is particularly good. There follows a discussion of the general principles of kinetics, including d'Alembert's principle and a first introduction to Hamilton's principle. This paves the way for the derivation of Lagrange's equations and the theory of small oscillations. One is rather surprised at the concise section on small oscillations, the discussion being often restricted to systems of two degrees of freedom.

The subsequent development of analytical mechanics due to Hamilton and Jacobi forms one of the most complete accounts of the subject. It includes a section on the application of Hamilton's principle to General Relativity.

Minimal principles so fecund in mechanics are also discussed at length, including statical principles such as Castigliano's theorem in elasticity.

* A useful bibliography is given on p. 5 of the book. Notable omissions are the text books by Lamb and Painlevé.

K

The section on rigid dynamics is remarkable for a complete discussion of the motion of the unsymmetrical top ; various methods of attack are discussed.

An elementary and useful discussion of the problem of n bodies is included with special reference to Lagrange's special cases for $n=3$ and $n=4$. The book closes with a chapter on non-holonomic systems and a discussion on dynamical principles given at a mathematics seminar at the Hochschule in Berlin.

It is hoped that some impression of the vast scope of the book has been conveyed by this notice. The examples worked out at the end of the book, which form the second part, are novel and interesting. There are few misprints and the book, which is beautifully printed and presented, will without doubt become one of the standard treatises on mechanics. V. C. A. FERRARO.

Wahrscheinlichkeit, Statistik und Wahrheit. By RICHARD VON MISES. Third edition. Pp. ix, 278. 31s. 1951. (Springer, Vienna)

This historically important book is already well known in this country through the English translation of the second German edition which was published in 1939 under the title of *Probability, Statistics and Truth* ; it has also been translated into Russian and Spanish. The appearance of a third German edition is doubly welcome because the English edition is now out of print and because the author has taken the opportunity of revising his work.

In 1919 von Mises began the attack on the foundations of Probability Theory which has interested Continental mathematicians for the past thirty years. His aim was to place Probability Theory on an axiomatic basis and to make of it a rigorous mathematical subject like geometry and mechanics. With the help of Kolmogorov and others this aim has been achieved, though not perhaps quite as von Mises expected. Expressed in the simplest terms, von Mises' thesis is that no satisfactory theory can be based on attempts to find the subjective " real " meaning of the term *probability* as it is commonly used ; it is necessary to stipulate a narrow definition of the term which is convenient for mathematical purposes in order to obtain a theory that is logical and coherent. Analogous situations occurred in mechanics in defining *work*, and, more recently, in the definition of *information* in Communication Theory, to mention two examples.

The book consists of six lectures on the definition, theory, and applications of probability expounded in simple non-mathematical language. The new edition is substantially the same as the second edition, though there are a few changes. There have been some amendments to the third, fourth, and fifth lectures : some of the more polemical sections have been omitted and their place has been taken by critical comments on subjects such as R. A. Fisher's " small sample " theory and his concept of " likelihood ", and on the contributions of Kolmogorov and Fréchet. The title misleads if it suggests that the book is a metaphysical treatise ; von Mises writes with the objectivity of an experimental scientist.

Though its original purpose has been achieved, and though some of von Mises' ideas have been superseded, the book remains a unique and readable discussion of the elements of Probability Theory and of a recent controversy by one of its protagonists. In view of the growing interest in the teaching of elementary statistics it is hoped that a new edition of the book in English will make its ideas more easily accessible in this country. B. C. B.

Analytische Geometrie. By F. NEISS. Mit 64 Abbildungen. DM. 9.60. Pp. viii, 167. 1950. (Springer, Berlin)

This text-book does not attempt to provide the training in manipulative technique at which an English work of the same standard would aim. What

it does provide is a careful introduction to the basic principles of the subject, presupposing some knowledge of linear algebra on which the author has written an earlier work. In five chapters the book provides an introduction to the coordinate geometry of the plane, and of ordinary space, including an account of congruent transformations, and to (mainly real) projective geometry in one, two and three dimensions, dealing with the principal properties of conics and quadrics and their metrical classification.

The reviewer is not quite clear that the distinction between real and imaginary points is treated with the same thoroughness as is accorded to other topics. The distinction between the two directions on a line, the definition of the angle between two lines, and the " normalisation " of direction ratios, all imply that the author's non-homogeneous coordinates are necessarily real ; yet points whose coordinates are not real seem to slip unintroduced into the story, and the index provides no obvious clue. This apart, the treatment of the subject is clear and concise and teachers will find in it many points of interest. The examples at the end of each chapter help to fill in questions of detail for which there is little room in the text.

The print and layout of the work is delightful ; every page is a pleasure to look at. D. B. S.

Colloque de Topologie (Espaces Fibrés). Pp. 137. 1225 fr. 1951. (Thone, Liege ; Masson, Paris)

This is the report of a conference held in Brussels in June, 1950, at which accounts were given of recent developments in the theory of fibre-spaces, by those European mathematicians working in this field of topology. Papers were read by H. Hopf, H. Cartan, Ehresmann, Koszul, Eckmann, Leray and G. Hirsch.

In an introductory paper, Hopf indicates how the study of fibre-spaces arose out of classical problems on the nature of the solutions of differential equations and outlines known results on the existence of continuous vector fields over varieties. Subsequent contributions deal with homology and homotopy relations between fibre-space, base-space, and fibre, some of the papers being concerned particularly with the application to the study of Lie groups. The algebraical situation arising in fibre theory is systematised, and this part of the conference is especially valuable for a proper understanding of the important recent work of H. Cartan, Leray, Koszul, Serre and others on filtered homology groups, and their application by Serre and Cartan to the calculation of the homotopy groups of spaces. The contribution by Eckmann provides a virtually complete account of the known results on the homotopy groups of fibre-spaces, including the homotopy criterion for the existence of a section when the base-space is a sphere.

Full references to research papers (and relevant chapters in Bourbaki), and comprehensive bibliographies further enhance the value of this excellent publication. P. J. Hilton.

Introduction to Statistical Method. By B. C. Brookes and W. F. L. Dick. Pp. viii, 288. (Part I obtainable separately.) 21s. 1951. (Heinemann)

The position of statistics in relation to mathematical teaching in schools has recently undergone a radical change through the introduction by some of the Examining Boards of optional statistics syllabuses both at the Alternative Ordinary and Advanced levels. Up to the present, statistics has enjoyed a very useful freedom, for it was a subject on which a few periods could be spent in an interesting and instructive manner by a class which had completed its examination requirements. Under such conditions the syllabus could be pleasantly elastic, matter which seemed to be lacking in appeal to the teacher

or the class could be left out or treated lightly and, in general, a very informal approach to the subject could be maintained. The change in status of the subject mentioned above inevitably means a change in the nature and intensity of teaching and in the text-books which are used to supplement that teaching.

The present volume does not pretend to restrict itself entirely to the syllabuses of these examining bodies. Its object, as stated in the preface, is to provide an introduction to statistical methods in the specialist study of branches of science such as physics, chemistry, biology, psychology and economics. But it is the first important text-book to appear since the change of conditions and it is therefore bound to be examined carefully from the standpoint of its suitability for school work.

It should be said at once that, in spite of some points to be mentioned later, this book will be welcomed warmly by mathematical teachers and that it will go a long way towards solving the question of a suitable text-book. Part I is a clear and straightforward exposition of the statistics of a single popualtion with good chapters on representation of numerical data, measures of position and dispersion, probability and the Binomial, Poisson and Normal distributions. Part II contains chapters on Tests of significance and sampling, Correlation and Regression, Goodness of fit and contingency tables and a final chapter on the planning of statistical experiments, including references to Latin squares and factorial design. The exposition is lucid and well-tabulated specimen examples, with clear explanations, are given throughout. In the greater part of the book the average intelligent Sixth-former could quite easily teach himself with occasional reference to a master—an important point in these days of short-staffing. There is a reasonable variety of examples drawn from diverse sources and a note in the preface, which might perhaps be given greater prominence, stresses the importance of the use of " live " material lying within the experience of the pupil. Rather more of the examples given might themselves have been chosen from such fields, though sometimes when an effort is made to do so the effect is a little odd. (It is, for example, rather difficult to imagine the next action of a pupil who turns over to p. 74 and finds that the next thing he has to do is to find the s.d. of 100 cricket scores collected by himself!)

The subject-matter is, in the main, very adequately covered. The section on Index numbers might perhaps be more detailed. There is, for example, no reference to the two ways of constructing such numbers (the " fixed-base " and " chain-base " methods). But the main criticism which may be made is that, from the point of view of the school pupil, some parts of the work are over-condensed. In dealing with the important idea of standard deviation, for example, a single worked example is made to illustrate the definition, the calculation of s.d. from an assumed mean (with a note on the supporting algebra), the calculation of s.d. from a frequency distribution by the mid-interval method, and the automatic check via $\Sigma f(d+1)^2$; and in addition to all this, the important result that the s.d. is least when measured from the mean is relegated to a note. This arrangement may very well be suitable for scientists or others using the book but it is not well-suited for school teaching purposes.

Somewhat similar considerations apply to the sections on correlation and regression. It would seem more practicable for this work to be arranged in two sections, an elementary section in Part I covering those ideas needed for the Ordinary Level syllabus, leaving to a later stage the examination of reliability and the question of the extent to which the correlation-statistics of a sample may do duty for the whole population. A good deal of the section on tests of significance and sampling lies outside the range of work likely to be

attempted in schools and here again it would be more useful if the elementary treatment of the distribution of sample means had been arranged in an earlier section, especially since the use of the formula $\sigma/\sqrt{n}$ is specified in the Ordinary Level of one of the examination syllabuses.

In the main the authors take the view that the mathematical background of the formulae quoted is too difficult for the reader of the book. For example, the development of the Normal distribution as a limiting case of the Binomial distribution, though hinted at in general terms, is not dealt with theoretically. This attitude occasionally produces a scrappy and rather uneven effect, for example, in dealing with degrees of freedom. When statistics was a "side-line" subject in schools the policy of suppressing the book-work was probably the right one and for the present-day student at the Alternative Ordinary level the same considerations probably apply. But it might reasonably be assumed that a student at the Advanced level would have sufficient acquaintance with pure mathematics to follow the majority of the proofs referred to, at any rate in an appendix.

In general, the presentation of the material is clear and pleasing and there are very few misprints. In Ex. 24 on p. 206 the phrase

$$z = k \,.\, \exp\, -(ax^2 + 2hxy + by^2)$$

looks odd at first sight, especially as the usual e notation is used elsewhere, and on p. 226,

$$z = \tfrac{1}{2} \log_e \frac{1+r}{1-r} = \tanh_r$$

is either a misprint or a very unfamiliar notation.

These, it should be emphasised, are very minor blemishes on an otherwise excellent production. This is a book which every intending teacher of statistics should examine carefully. It is certainly worth a place in every school library. G. L. P.

Applied Mechanics for Engineers. By Sir Charles Inglis. Pp. xii, 404 42s. 1951. (Cambridge U. P.)

Never have I enjoyed reading a book on applied mechanics as much as the present volume. In fact, in the past I have always been wary of books with applied mechanics in the title as they are generally associated with dull arithmetical calculations on such topics as pulleys, etc. In my opinion, there are no tiresome topics in this volume, although the book is certainly practical enough. In fact, it is based on lectures given in the past to students reading for an honour's degree in engineering at Cambridge. However, there is plenty of analysis to keep the more theoretically minded mathematician interested.

There are twenty chapters in all, ranging from the fundamental principles of rigid body statics to gyroscopic principles and applications. Some of the topics covered, in addition to the more usual ones, are theory of taut wires, deformation of frameworks, coupled vibrating systems, aperiodic springs, pressure waves and self-excited oscillations. All examples, of course, are taken from actual engineering problems, whether the topic be friction or kinetic energy. For example, we have the increased grip produced by a V-notch, or the gear ratio to give maximum acceleration to a motor car travelling at a given speed along a level road, together with hundreds of other interesting applications to applied mathematical principles. As can well be imagined too, the author has quite a lot to say about frameworks, both analytical and graphical.

I feel sure most applied mathematicians, who are not engineers, will also be most interested in this book, if only to see how their subject is adapted and

extended by engineers. Teachers of the subject should also read this book, for I have no doubt that it will help them to make their lectures more interesting to their students. Again, I heartily recommend the book to all students who are thinking of becoming mathematicians in Industry.

In conclusion, I should add that there are numerous examples set for the reader, many of which are taken from past examination papers. The printing is excellent and the diagrams are finely executed and interesting in their detail.

JOHN WILLIAMS.

Mechanics. By S. BANACH. Pp. iv, 546. $6.00. 1951. Monografie Matematyczne Tom xxiv (Warsaw). Translated from Polish into English by E. J. Scott. (Stechert-Hafner, New York)

The present volume is written by a mathematician who needs no introduction to anyone who professes to be interested in the subject. In fact, it is not often we get the opportunity of reading an elementary textbook written by a famous mathematician. When an opportunity such as this occurs, it is to our advantage to study the book carefully for we are certain to learn something new. This book is no exception to the rule.

The substance of the work is derived from lectures delivered by the author at the Jan Kazimierz University and Lwów Polytechnic Institute and covers the whole range of mechanics (dynamics and statics) from first principles to Hamilton's equations and the principle of least action. Vectors are used wherever convenient throughout the book, and consequently, the first chapter is devoted to the basic operations on vectors and ends with a short account of the reduction of systems of line vectors.

Chapters II and III deal with the kinematics and kinetics, respectively, of a particle. As may well be imagined, this latter chapter covers a large ground and is divided by the author into two sections. These sections deal with the dynamics of an unconstrained point and a constrained point. In the first section, we find such sub-headings as Newton's laws, central motion, work, energy, etc., whilst in the second, we find paragraphs devoted to the constrained motion of a point along a curve, the pendulum, and finally, motion relative to the earth. The most interesting feature of these chapters is the beautifully lucid account of the force of Coriolis, a topic all too frequently ignored by many writers on mechanics.

Chapter IV, as a prelude to the dynamics of systems of particles and rigid bodies, develops the theory of centres of mass and moments of inertia both of systems of particles and continuous distributions. Chapter V, on the other hand, is concerned with the dynamics of a system of discrete particles.

The statics of the volume is developed almost entirely in Chapter VI and is divided into three sections dealing in order, with the unconstrained rigid body, the constrained body and systems of bodies. The first section gives an account of the various methods of resolving forces, etc., whilst the second and third sections deal respectively with passive forces (friction, etc.) and frameworks, etc.

The remaining five chapters are ones which are of a more advanced nature than the preceding, but are written in such a logical and lucid style that their reading is straightforward. In fact, I particularly recommend these chapters to all students reading for honours in mathematics. The titles of these chapters are : VII. Kinematics of a rigid body ; VIII. Dynamics of a rigid body ; IX. Principle of virtual work ; X. Dynamics of Holonomic systems ; XI. Variational principles of mechanics. I found the chapter on virtual work particularly interesting and it gives the reader a thorough introduction to Lagrange's equations and the subsequent work.

In my opinion, the book is a valuable contribution to the existing textbooks

on mechanics. All through the volume one gets the feeling that the work is the product of a master of the subject. There are numerous well-chosen examples worked in the text to illustrate or amplify various points of theory though none is set for the reader.

The printing is excellent and the numerous and helpful diagrams neatly executed. The only fault I have to find with the work is that the continental binding is certainly not adequate for such a large volume.

In conclusion, a few words of praise and our grateful thanks (*i.e.* from those of us who cannot read Polish) are due to Prof. Scott for the labours of translation.

JOHN WILLIAMS.

Space, Time and Matter. By H. WEYL. Translated from the fourth German edition by H. L. BROSE. Pp. xiii, 330. 1951. (Dover Publications, Inc.)

The attention of students of relativity theory should be drawn to this re-issue. It represents the first American printing of the 1920 edition of the book, and is carried out with the accomplishment which one has learnt to expect from Dover Publications. In his (1950) preface, Weyl recalls that the aim of classical physics is to construct a field theory from which all forces of nature can be derived by uniform methods, and admits that his principle of " gauge invariance " has not proved to be as successful in this attempt as he had hoped. For more recent developments the reader is referred to Schrödinger's *Space-Time Structure* which was published last year by the Cambridge University Press (apparently announced by Macmillan in U.S.A.)

P. T. LANDSBERG.

Basic trigonometry. By C. O. TUCKEY. Pp. ix, 204. 6s. 6d.; with answers, 7s. 6d. 1951. (Christophers)

This book takes the beginner to a stage where most of the trigonometry required for the Advanced Level of the General Certificate of Education has been covered. The candidate for Scholarship papers will need to go beyond the scope of this book.

The author brings great freshness to his treatment of the subject, and his choice of topics and examples is to be highly commended. The first five chapters in the book cover, approximately, the Ordinary Level syllabus (including Additional Mathematics) and chapters VI to IX deal with Advanced Level work, and much work which is not included in most examination syllabuses, but which is of great mathematical interest.

The sine is the first trigonometrical ratio to be discussed. Having solved problems on the right-angled triangle, using sines, the author proceeds straight to the sine rule for any triangle. A similar procedure is followed in the next chapter with the cosine and the cosine rule. Not until page 50 of the book is the tangent mentioned. Meanwhile, examples such as : " In a triangle two angles are 36·9° and 53·1° and the smallest side is 18″ ; find the side opposite 53·1 " are done using the sine rule. A sine table (without differences) is provided. Thus the teacher is almost compelled to follow the order of treatment laid down by the author.

In the elementary part of the book there is frequent reference to three-dimensional work, and there are numerous sets of easy examples, with revision exercises at the end of the chapters.

Great care is taken in promoting systematic methods of calculation in the numerical solution of triangles. On this topic there are interesting sections on the calculation of the distance between two observed points (involving the solution of three triangles), Air Navigation, and some practical surveying problems.

The last three chapters (occupying approximately half the book) cover the following topics :

Chapter VII. Small and large angles, radian measure, the addition formulae and the solution of trigonometrical equations. No general proof of the addition formulae is attempted, and the " sum and product formulae " are nowhere mentioned.

Chapter VIII. Calculation of tables, projections and oblique sections, vectors and vector products, complex numbers (called ordered pairs) and trigonometry on the sphere. In this chapter the work is too condensed, and the treatment of complex numbers, especially, so slight that the reader has very little chance of developing any technique for using the ideas introduced.

Chapter IX. " Topics of trigonometrical interest ", especially projectiles, graphical solution of equations, some pure geometry disguised as trigonometry, a simple nomogram for solving

$$1/f = 1/v + 1/u,$$

sundials, π, plan and elevation, harmonic motion, polar coordinates, cycloids and allied curves, diagrams for the solution of triangles.

It may be held that in breaking away from the more usual formal treatment of the subject the author has overdone the idea of " topics ". His matter and its development are interesting and often most attractively displayed. The teacher will need to supplement much of the work in the last two chapters with a good many notes and extra examples.

The type used is clear but is often cramped in appearance. Diagrams in general are poorly done and have minute lettering that is very hard to read.

F. J. TONGUE.

The Adversaries. By J. SOMMERFIELD. Pp. 376. 15s. 1952. (Heinemann)

The review of a novel in the *Gazette* calls for an explanation, which may be given by observing that the mathematician is usually treated as a figure of fun by the novelist, whereas Mr. Sommerfield has given an honest picture of a great mathematician. The story is based on the life of Galois, but is frankly fiction and does not pretend to be that unsatisfactory compound, the " imaginative reconstruction ". The theme is the three-fold cord of mathematics, politics, love-affairs which wove Galois' tragic fate, and mathematics is regarded as a vital and natural thread in the story, just as music might be so regarded in the life of a great musician—not the whole of life, not entirely divorced from life, but a sensible and indeed natural constituent.

The style is frankly and even crudely realistic. Perhaps this is why the description of the barricades and street-fighting in Paris in 1830 is convincing in its squalors and heroisms.

Mathematicians may thank Mr. Sommerfield for a sincere application of his art to theirs.

T. A. A. B.

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